

Non-abelian fundamental groups in arithmetic geometry: final report

1. Main Themes

The ancient problems of diophantine equations and diophantine geometry are still absolutely central in research in number theory today, and the programme attempted to explore some of the most promising avenues of attack on these problems. It was primarily concerned with the two themes of anabelian geometry and non-commutative Iwasawa theory, and the mathematical ideas underlying these subjects. More specifically, much of the programme was devoted to the following topics:

- (i) the section conjecture of Grothendieck;
- (ii) the main conjectures of non-commutative Iwasawa theory;
- (iii) the applications of non-abelian fundamental groups to the study of multiple zeta values and rational points on curves on hyperbolic curves.

The ubiquity of the section conjecture was made clear because of its connexion with such diverse themes as the Brauer-Manin obstruction, class field theory, and the arithmetic theory of moduli. Connexions between themes (ii) and (iii) were developed throughout the programme. We wish to emphasize that such connexions are very surprising because (ii) is primarily motivated by the celebrated conjecture of Birch and Swinnerton-Dyer relating the group of rational points and the Tate-Shafarevich group of an abelian variety over a number field to the behaviour of its complex L-function at $s=1$, whereas, until recently, the study of rational points on hyperbolic curves had seemed to have little to do with L-functions and their special values. We hope that the programme made a large group of researchers from around the world much more familiar with these highly promising ideas. Indeed, it may well be that current research on these problems is somehow parallel to the situation at the end of the nineteenth century, when the reciprocity law of class field theory was emerging in a number of seemingly different and unrelated forms, and the fundamental unity of the subject only became clear in the 1920's with Artin's discovery

2. Programme Structure

There was a loose division of the activities into the areas of anabelian geometry and non-commutative Iwasawa theory, with many of the boundaries being left intentionally unclear. In all, there was one workshop concerned with Iwasawa theory, one with anabelian geometry, a Spitalfields day on potential modularity, and two workshops where a conscious effort was put in to bring together people of diverse specialities. The workshops were supplemented by intensive lecture series by Pierre Deligne, Hiroaki Nakamura, Florian Pop, Mohamed Saidi, Christophe Breuil, Peter Schneider, John Coates, Ramadorai Sujatha, Mahesh Kakde, and Minhyong Kim.

In between the longer-term activities, there was a series of seminar lectures delivered by P. Deligne, U. Jannsen, P. Lochak, L. Schneps, Z. Wojtkowiak, D. Blasius, B.-D. Kim, M. Antei, C. Popescu, Y. Hachimori, M. Shashahani, and A. Wiles, as well as numerous study sessions organized by M. Kakde on Thursday afternoons.

Workshops

(1) Opening workshop, 27–31 July

The organizers were John Coates, Minhyong Kim, Richard Taylor, and Andrew Wiles. A broad range of experts on motives, automorphic forms, and Galois representations were present in addition to a corps of specialists on anabelian geometry and Iwasawa theory. To an extent, the goal was to survey the different investigations to which the themes of the programme could be expected eventually to contribute, as well as to advertise its main ideas in relatively concrete form to the arithmetic community at large.

(2) Anabelian geometry workshop, 24–28 August

The organizers were Minhyong Kim, Florian Pop, and Mohamed Saidi. The talks surveyed various perspectives on Grothendieck's section conjecture and its ramifications, including the theory of the

Brauer-Manin obstruction, reconstruction theorems for fields and schemes, as well as connections to Diophantine geometry and logic.

**(3) Workshop on non-commutative algebra and non-commutative Iwasawa theory
(Satellite meeting at ICMS Edinburgh), 28 September–2 October**

The organizers were Ken Brown, David Burns, John Coates, and Peter Schneider. The aim of the workshop was to discuss related recent developments in non-commutative algebra and Iwasawa theory. Iwasawa theory, via its main conjectures, remains the only systematic method known today for studying the mysterious connexions between purely arithmetic problems and special values of complex L-functions, typified by the conjecture of Birch and Swinnerton-Dyer. There is ever growing evidence that these main conjectures hold in vast generality, in particular, for all motives over p-adic Lie extension of number fields. At the same time, the Iwasawa algebras of compact p-adic Lie groups provide one of the most interesting examples of non-commutative rings. The workshop had lectures by both number-theorists and experts in non-commutative algebra, aimed at explaining to each other their results, conjectures, and hopes for future work.

(4) Spitalfields day on potential modularity, 30 October

The organizers were John Coates, Tim Dokchitser, Vladimir Dokchitser, and Mohamed Saidi. The theorems of R. Taylor and collaborators on potential modularity have resulted in dramatic recent progress on long-standing problems of number theory, including a proof of the Sato-Tate conjecture and important cases of the Fontaine-Mazur conjecture. The goal of this workshop was to explain the techniques involved in the key results to students and to number-theorists without particular expertise in automorphic forms.

(5) Closing workshop, 14–18 December

The organizers were David Burns, John Coates, Guy Henniart, Minhyong Kim, Florian Pop, and Mohamed Saidi. In addition to talks on anabelian geometry and Iwasawa theory, there were a number of lectures by top experts in rational points on varieties, p-adic representations, special values of complex L-functions, automorphic forms, and the interactions between these areas. The hope was to provide not only a picture of the questions/problems discussed/solved during the programme, but also to flesh out future possibilities for applying non-abelian techniques to a broad range of arithmetic investigations.

Lecture Series

(1) P. Deligne: Multiple zeta values, 10, 17, 21 August

Much of the interest in arithmetic fundamental groups of the recent decades stems from a seminal paper written by Deligne in the 80's, where periods and Galois theory were studied for the unipotent fundamental group of the projective line minus three points. The numbers that arise there as periods are the *multiple zeta values*, which are expected to encode essential information about the category of mixed Tate motives. In these lectures, Deligne surveyed the state of the art on the ring of multiple zeta values, including connections to Grothendieck-Teichmüller theory and recent results of Furusho.

(2) F. Pop: Anabelian Geometry I-IV, 4-7 August

The lectures focused on the so called (anabelian) Bogomolov's Program, which has as final aim the reconstruction of function fields K/k of varieties of dimension > 1 over algebraically closed base fields k from the pro-abelian-by-central Galois group Π_K of K . One of the main points of the lectures was to present the "local theory" with the several subtleties of the valuation theory of K , and the way they are reflected in Π_K . Based on this, we finally showed how one can *functorially* reconstruct K/k from Π_K in the case k is an algebraic closure of a finite field, thus completing Bogomolov's Program in this case.

(3) H. Nakamura: Anabelian Geometry V-VIII, 11-14 August

These lectures described results of the lecturer and many others on the centralizer of the outer Galois action on the various arithmetic fundamental groups of hyperbolic curves, especially the question of when these centralizers are spanned by geometric automorphism. Many interesting results on the profinite and pro- l versions were described, and a number of interesting questions posed for the case of unipotent fundamental groups.

(4) M. Saidi: Anabelian Geometry IX-XII 18-21 August

These lectures described recent results of the lecturer and Akio Tamagawa on how to reconstruct hyperbolic curves over finite fields from their geometrically pro- Σ arithmetic fundamental groups for a set of primes Σ not containing the characteristic of the base field and finitely many primes. It also gave an overview on the anabelian geometry of algebraic curves over algebraically closed fields of characteristic $p > 0$ by discussing some conjecturally expected statements and ideas on how to investigate these conjectures.

(5) C. Breuil: Representations of Galois and of GL_2 in characteristic p , 7, 8-10 September

This was a course on the p -adic and mod p local Langlands correspondence, with a rather careful discussion of the results of Colmez on $GL_2(\mathbb{Q}_p)$, and some intriguing proposals for the difficult problem of extending the correspondence to finite extensions of \mathbb{Q}_p .

(6) J. Coates and R. Sujatha: Iwasawa theory of elliptic curves over p -adic Lie extensions of dimension greater than 1, 15-17, 18, 22-25 September

The lectures surveyed what is known about the $M_H(G)$ -conjecture, which is of central importance in non-commutative Iwasawa theory, especially for the formulation of the main conjectures of non-commutative Iwasawa theory.

(7) P. Schneider: The algebraic theory of p -adic Lie groups, 1-3, 8-10, 15-17, 22-24 September

The fundamental object of noncommutative Iwasawa theory is the completed group ring (or the Iwasawa algebra) of a Galois group which also is a p -adic Lie group. Lazard in his seminal paper in 1965 has developed a technique to analyze the structure of these rings (proving, for example, that they are Noetherian). In fact his approach was axiomatic through the notion of a p -valuation on a pro- p group. This course gave a self-contained and complete treatment of Lazard's theory.

(8) M. Kim: Motivic fundamental groups and Diophantine geometry, 5,7, 12, 16 October

These lectures gave an introductory survey of the applications of non-abelian fundamental groups to the arithmetic of hyperbolic curves. In particular, the construction of Selmer varieties and unipotent Albanese maps were carefully discussed.

(9) M. Kakde: The main conjectures of Iwasawa theory 9, 16, 18 November

The main conjectures of Iwasawa theory propose that arithmetically important modules for the completed group algebras of (essentially) p -adic Galois groups can be analyzed using suitable p -adic L -functions. The efforts of Coates-Fukaya-Kato-Sujatha-Venjakob have revealed that the key issue is to construct canonical compatible trivializations of the determinants of compact support étale cohomology for arithmetic sheaves over number rings. This series of lectures presented the basic formalism of the main conjectures in a form that made the transition from the commutative to the non-commutative case transparent and uniform, and outlined the lecturer's results for totally real fields.

3. Outcome and achievements

Since the previous section has included already a fair amount of detail on the activities, we will give here just a brief summary of the outcome and developments.

The programme attracted over 80 visitors for varying lengths of time from many parts of the world in addition to the speakers and participants of the workshops. Following the plan initially proposed, many people with no regular contact in other venues were brought together. Experts in the theory automorphic forms, such as Clozel, Henniart, and Blasius, expressed much appreciation about the opportunity to attend lectures on anabelian geometry and Iwasawa theory, and even Deligne felt that he was brought up to date on the latest developments in the theory of fundamental groups through the lecture series and workshops during his month-long visit. Many people working on fundamental groups were pleased with the opportunities to learn the intricate techniques of Iwasawa theory, both at the forefront of research as in the lectures of Coates and Sujatha, and in the foundations, as in the systematic series by Schneider. Conversely, those who work in Iwasawa theory were able to learn about current important developments in the arithmetic theory of fundamental groups

Workshops

All of the workshops constituted solid contributions to the the programme and received much attention both in the UK and abroad. The organizers received a substantial number of reports on the positive impact of the lectures through internet dissemination as well, which were perceived to be of very high quality. The opening workshop was especially successful, with many of the leading researchers in the world in arithmetic geometry and the theory of automorphic forms present. The lectures surveyed many promising current ideas of arithmetic geometry, which were broadly related to the main themes of the programme. It was also very effective in advertizing the programme themes to a wide audience of number-theorists.

Collaborations

Continuation of previous collaborative projects were ubiquitous, such as Coates-Sujatha, Cadoret-Tamagawa, Pop-Stix, Gangl-Nakamura-Schneps, Shin-Park, Saidi-Tamagawa, Ardakov-Wadsley. Many new projects were catalyzed as well by the activities of the programme, as in joint work of Stix-Ciperiani (divisible elements in Tate-Shafarevich groups), Stix-Vdovina (arithmetic geometry and geometric group theory), Harari-Voloch (Skolem's problem and Brauer-Manin obstructions), Colliot-Thelene-Swinnerton-Dyer (unramified cohomology of cubic threefolds), and Antei-Saidi (solvable torsors for fundamental group schemes).

Of course, many discussions that have not yet led to concrete collaborations were cited by several visitors as important stimuli to their research. Andrew Wiles, for example, took the opportunity of his visit to learn about fundamental group techniques in Diophantine geometry, and the visit of Deligne made a major impact, his strong influence being acknowledged by by Ardakov, Abrashkin, Besser, Hadian, and Gangl. Just after the end of the programme, an interesting development occurred in the work of participant Yuichiro Hoshi, who managed to construct counterexamples to the pro- p analogue of the section conjecture. This refined considerably our understanding of the mathematical subtleties surrounding Grothendieck's conjecture and brought to the fore the natural question of non-abelian local conditions appropriate to the pro- p section conjecture.

Publications

In addition to a number of research publications that came out of the institute's activities, a volume of proceedings has been proposed to Cambridge University Press, whose approval process is in its final stage. All the organizers of the special programme will participate as editors, and articles have been contributed by or solicited from Pop, Nakamura, Saidi, Schneider, Kakde, Coates, Sujatha, Kim, Buzzard, Breuil, Emerton, and Calegari. It is expected that this will be the first publication in which the study of arithmetic fundamental groups will be surveyed in conjunction with Iwasawa theory and the various aspects of the Langlands' programme.

One other major publication that can be said to have been influenced by the programme is the book by Schneider on Lazard's theory of p -adic Lie groups. Preparation of the lecture series appears to have contributed greatly to furthering the completion of the manuscript.

Future developments

Several follow-up events are already in the planning stages, such as an anabelian component in a proposal for the Durham symposium on Galois representations in 2010, and a proposal for a special trimester on anabelian geometry, Diophantine geometry, and automorphic forms at the Hausdorff institute in Bonn.