

# Spaces of Kleinian Groups and Hyperbolic 3-Manifolds

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Report from the Organisers: Y Minsky (Stony Brook), M Sakuma (Osaka),  
C Series (Warwick)

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The modern subject of Kleinian groups and hyperbolic 3-manifolds divides roughly into three phases: the Ahlfors-Bers school (before about 1980), the Thurston era (1980-95), and post-Thurston. Thurston's ideas, which centred on showing that very many 3-manifolds can be endowed with a hyperbolic structure, revolutionised the field. After more than a decade of turmoil spent debating and consolidating his insights, by the mid-'90s we appeared to have settled down on a number of hard problems which still seemed pretty far out of reach. The last two or three years, however, have seen a series of profound and far reaching breakthroughs, with new results and techniques appearing almost monthly, to the point where many of these old problems are now close to resolution.

This meeting turned out to be the international gathering at which this plethora of new results was disseminated to a wider audience for the first time. Almost all the primary contributors took part, among them a remarkable group of young people who between them are leading much of the advance. Quite how rapid progress has been only became apparent to most of us during the meeting.

Listing individual talks and publications cannot fully do it justice; the programme will be remembered as the moment at which all of these new ideas were brought together, laying a common foundation for future work.

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## Problems and Solutions

The programme was especially concerned with the classification and location of discrete groups in the ambient space of all representations from the fundamental group of some fixed 3-manifold into  $SL(2, \mathbb{C})$ . Three major problems were open when the meeting was first planned: Marden's Tameness Conjecture, the Bers Density Conjecture and Thurston's Ending Lamination Conjecture. Of these, the last two have now been proved for all tame groups and there has been a huge extension in the class of groups known to be tame.† Much of our activity naturally centred round understanding this flurry of work.

The simplest class of Kleinian groups is the geometrically finite ones for which there is a finite sided fundamental polyhedron; the deformation theory of these groups was worked out by Ahlfors, Bers and Marden and largely understood pre-Thurston. A hyperbolic 3-manifold is called tame if it is homeomorphic to the interior of a compact 3-manifold. Geometrically finite groups are necessarily tame. Marden conjectured in the '70s that all (finitely generated) hyperbolic 3-manifolds are tame. Major progress has been made in joint work of Brock, Bromberg, Evans and Souto, culminating in a proof by Brock and Souto that all algebraic limits of tame groups are tame. A corollary is the Ahlfors measure conjecture for tame

groups: the limit set has either full or zero Lebesgue measure.

Ahlfors-Bers theory shows that geometrically finite groups are determined by the analytic (Teichmüller) data of their ends. Thurston and Bonahon showed in the '80s that if the 3-manifold has incompressible boundary, then each non-geometrically finite end contains a sequence of closed geodesics which exit the end and which limit on an 'ending lamination', from which it can be shown that the end is tame. The Ending Lamination Conjecture (ELC) asserts that a hyperbolic 3-manifold is completely determined by these 'end invariants': the analytic data of each geometrically finite end and the ending lamination of the rest. In the last year Brock, Canary and Minsky have completed the proof of the ELC for all tame hyperbolic 3-manifolds. The proof proceeds by decomposing the manifold into 'blocks' whose arrangement is determined from the end invariants by a certain path through the curve complex. This is a simplicial complex that encodes the combinatorial structure of the set of simple loops on a surface. This path is shown to control the arrangement of short geodesics, hence the geometry of the blocks, from which it is possible to reconstruct the manifold up to quasi-isometry.

The Bers Density Conjecture states that every finitely generated Kleinian group can be approximated by geometrically finite ones. Contributions from many people, including most recently Brock, Bromberg, Canary, Evans, Kleineidam, Ohshika and Souto, have culminated in a proof of this conjecture for all tame groups. A crucial step was an innovative approximation technique introduced by Bromberg. Together with many other recent developments, this rests on the deformation theory of cone manifolds. Initiated by Kerckhoff and Hodgson in the mid-'90s, this technique has been honed into a powerful tool. Recent refinements allow one to construct and control deformations of groups in very precise ways. Another important contribution was a new criterion for the existence of algebraic limits due to Kleineidam and Souto.

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## Structure of the Programme

Aside from the very intense Euroconference which took place in the third week, the programme was kept fairly unstructured to allow maximum time for informal interaction between participants. Many commented on just how ideally suited the architecture of the Institute is to this purpose. From early to late, groups of participants were to be heard in lively conversation on all sides.

Organised seminars, especially in the second week, gave all participants who wished the opportunity to speak outside the workshop. Since the conference programme was necessarily very crowded, this was much appreciated, and several additional talks were given by popular request. The final week was especially valued by those who were able to stay on, producing an interesting mix of people and some unexpected interactions.

A much commented on and appreciated feature was the mix of participants from the different schools in Europe, the USA and Japan. Many people met their overseas counterparts for the first time and numerous new transcontinental collaborations were initiated. We shall be pleased if this leads to future enrichment and evolution of the field.

The several computational experts (Dumas, Wada, Wright, Yamasita) made a considerable contribution, producing new computations and graphics 'to order' in response to the questions and conjectures of participants. Their graphics programs received much wider explanation and publicity than hitherto and one can anticipate that this will be very beneficial for future developments.

## **Spaces of Kleinian Groups and Hyperbolic 3-Manifolds Euroconference, 3-8 August 2003**

This conference was extremely successful and could not have been better timed. We were over-subscribed and in the event exactly 100 people took part, among them almost all the main contributors to recent research. There was a very good mix of ages from the most senior figures to new graduate students.

There was a full programme of talks which maintained a notably high standard. Speakers included a substantial number of very talented young people; others presented their work in two lively poster sessions. Canary and Minsky between them outlined the proof of the ELC while the talks of Bromberg, Evans and Souto centred on the Density Conjecture and limits of tame groups. Kerckhoff explained some refinements of deformation theory which are important for many applications.

A theme common to many of the talks, especially those of Souto, Miyachi and Ohshika, was the topological and analytic machinery needed to extend results from the incompressible boundary case to all tame groups. Often building on the new results, we now have a much better understanding of the topology of deformation spaces, another of the field's important open problems. For example Holt presented results about the rather mysterious 'bumping' between various components of the discreteness locus.

Another conjecture of Thurston is that a Kleinian group is determined by the geometry of its convex core boundary. While important questions remain, Lecuire has recently completely characterised the possible bending laminations. In the case when the bending lines are closed curves, Series and Choi used cone manifold theory to show that the traces of the bending lines are local parameters for the deformation space. This is a significant step towards obtaining pictures of the discreteness locus in higher dimensional deformation spaces.

Another theme was the combinatorial structure of Teichmüller spaces, mapping class groups, and the complex of curves on surfaces. This last played a crucial role in the proof of the ELC. Hamenstädt and Bowditch in particular gave beautiful talks discussing their results which simplify and extend important earlier results of Masur and Minsky. Hamenstädt's results actually go much further by analysing the quasi-isometry problem for the mapping class group and solving the "rank conjecture" for quasi-flats in the group. Masur, Rees and Wolpert all discussed various closely related aspects of the geometry of Teichmüller space and the complex of curves.

A somewhat novel feature, which met with participants' approval, was that the first day was devoted mainly to experimental computer graphics. Such graphics have provided insights and underpinned or inspired much recent research, notably concerning the shape and topology of deformation spaces. Several talks explaining algorithms led to lively discussions, with new ideas actually being implemented during the conference. One such was the first ever picture of 3-dimensional slices of the discreteness locus in a 4-dimensional parameter space, made using software developed for medical imaging.

It was a packed week, with a conference dinner in Emmanuel College and a very enjoyable Book Evening hosted by Cambridge University Press. In place of the traditional T-shirts, we celebrated the input of computer graphics by distributing a set of six postcards of Kleinian group pictures, sponsored by Cambridge University Press.

Participants will remember the first week of August as the hottest yet on record; the success of the conference can be in part measured by the fact that the lecture room remained full to the end despite the almost overwhelming heat.

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## Outcomes

Participants agreed that this programme will be remembered as a definitive moment for the field. It brought together almost all the major players and introduced the young people who have recently made such major contributions to the wider community. It played a crucial role in disseminating the torrent of new ideas and facilitated numerous new contacts between key researchers. There is no doubt that it will have set the agenda for the next phase of research. By the end of the meeting, there was a general realisation that the post-Thurston subject has fully come of age.

Aside from the numerous individual publications and collaborations which will result, there will be conference proceedings edited by the three organisers. This will be a companion volume to *Kleinian Groups and Hyperbolic 3-Manifolds*, the Proceedings of the Warwick workshop, September 2001, which has recently appeared as LMS Lecture Notes 299 (2003). We conclude with a few comments from participants:

“I have attended many conferences in Kleinian groups, but this has been the best, the most comprehensive, and stimulating, that I have attended.”

“The main purposes of the programme were to reveal, to the broader community, the recent ground-breaking techniques, results and experimental work in the area. In this regard, it succeeded beyond my wildest expectations.”

“It was a brilliant idea to organise this workshop. Thank you very much to the Newton Institute and all the people who helped us during our wonderful stay in Cambridge.”

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