Moments of the Riemann Zeta Function and Random Matrix Theory

Chris Hughes

Overview

- We will use the characteristic polynomial of a random unitary matrix to model the Riemann zeta function.
- Using this, we will present a conjecture for the moments of the zeta function.
- We will use this conjecture to predict extremely small values of the zeta function.
- This will naturally lead into the study of discrete moments of the derivative of zeta.
- We will use random matrix theory to make a conjecture about such moments.

The Riemann zeta function

For $\mathfrak{Re}(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \,.$$

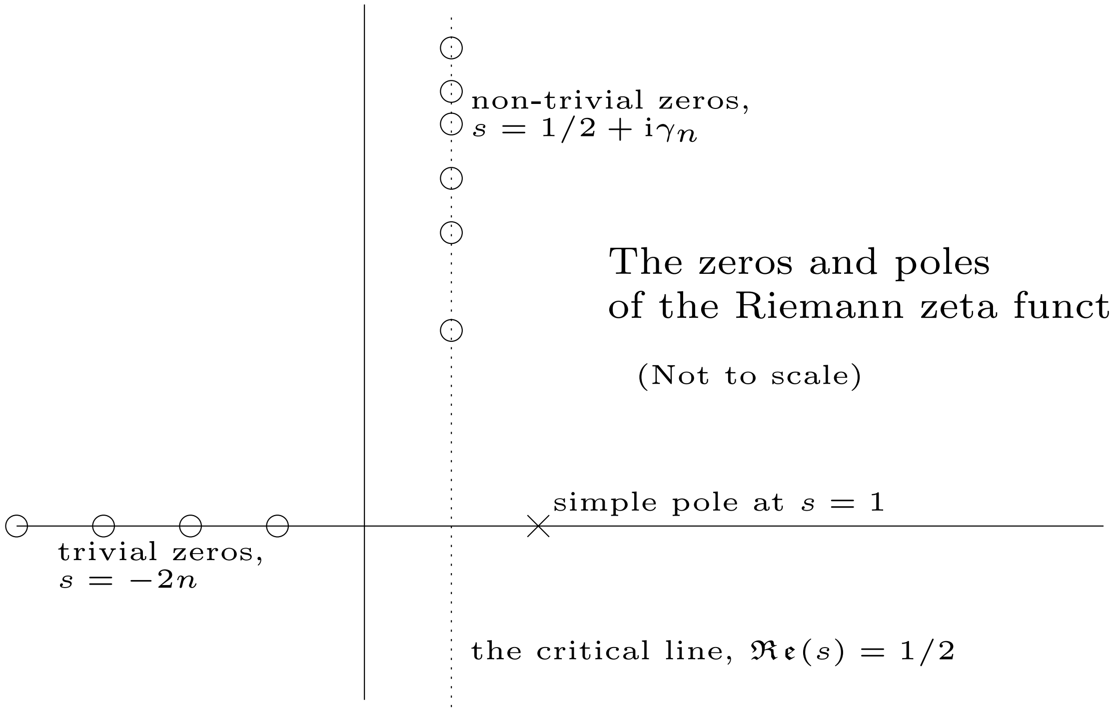
Analytic extension into the whole complex plane, apart from a simple pole at $s = 1$

Functional equation:

$$\zeta(s) = \chi(s)\zeta(1 - s)$$

where

$$\chi(s) = \pi^{s-1/2} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}$$



Number of zeros in critical strip:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \mathcal{O}(\log T)$$

If ρ is a zero, so is $1 - \rho$, $\overline{\rho}$, $1 - \overline{\rho}$.

Riemann Hypothesis: $\mathfrak{Re}(\rho) = \frac{1}{2}$

Under RH, $1 - \rho = \overline{\rho}$.

Random unitary matrices

Let $\mathcal{U}(N)$ be the set of $N \times N$ unitary matrices. It is a compact Lie group.

This means it has a (unique) Haar measure (left- and right-invariant measure). That is for any measurable set $\mathcal{A} \subset \mathcal{U}(N)$ and any $U \in \mathcal{U}(N)$,

$$\mathbb{P}_{\text{Haar}}\{\mathcal{A}\} = \mathbb{P}_{\text{Haar}}\{U\mathcal{A}\} = \mathbb{P}_{\text{Haar}}\{\mathcal{A}U\}$$

H. Weyl calculated the probability density of eigenangles under Haar measure. It equals

$$\frac{1}{(2\pi)^N N!} \prod_{1 \le j < k \le N} \left| e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k} \right|^2$$

where $\theta_1, \ldots, \theta_N$ are the eigenvalues.

Take an $N \times N$ unitary matrix, and scale its eigenangles by $\frac{N}{2\pi}$, so they have average spacing unity.

Scale the non-trivial zeros around height T by $\frac{1}{2\pi}\log\frac{T}{2\pi}$, so they have average spacing unity.

There is much evidence (theoretical, heuristic and numerical) relating statistics of the scaled zeros of the Riemann zeta function high up the critical line, with statistics of the scaled eigenangles of large Haar-measured unitary matrices.

This led Keating and Snaith to model the Riemann zeta function with the characteristic polynomial,

$$\begin{aligned} Z(U,\theta) &:= \det(I - Ue^{-\mathrm{i}\theta}) \\ &= \prod_{n=1}^{N}(1 - e^{\mathrm{i}(\theta_n - \theta)}) \end{aligned}$$

of an $N \times N$ random unitary matrix, U, with eigenvalues $e^{\mathrm{i}\theta_n}$.

Equate density of eigenangles with density of zeros:

$$N = \log \frac{T}{2\pi}$$

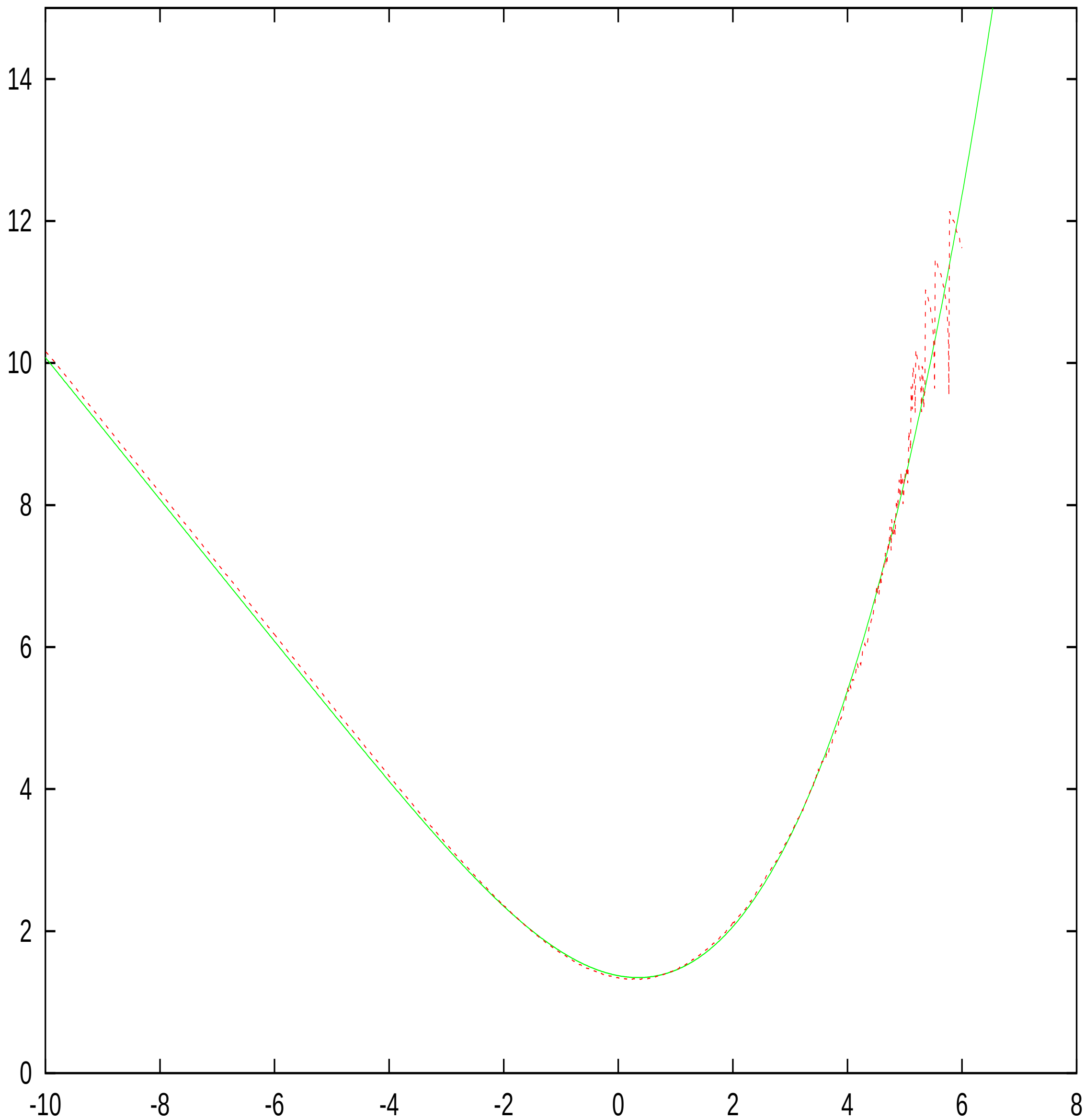
Central limit theorems

Theorem (Selberg). *For rectangles $B \subseteq \mathbb{C}$,*

$$\frac{1}{T}\,\mathrm{meas}\left\{T \le t \le 2T \;:\; \frac{\log\zeta(\frac{1}{2}+\mathrm{i}t)}{\sqrt{\frac{1}{2}\log\log T}} \in B\right\}$$
$$\to \frac{1}{2\pi}\iint_B e^{-(x^2+y^2)/2}\,\mathrm{d}x\,\mathrm{d}y.$$

Theorem (Keating and Snaith). *For fixed θ,*

$$\mathbb{P}_{\mathrm{Haar}}\left\{\frac{\log Z(U,\theta)}{\sqrt{\frac{1}{2}\log N}} \in B\right\}$$
$$\to \frac{1}{2\pi}\iint_B e^{-(x^2+y^2)/2}\,\mathrm{d}x\,\mathrm{d}y$$



Graph of the negative log of the value distribution of $\log|\zeta(\frac{1}{2}+\mathrm{i}t)|$ around the 10^{20}th zero (red), against the negative log of the probability density of $\log|Z(U,0)|$ with $N = 42$ (green).

Set $\sigma := \sqrt{\frac{1}{2}\log N}$. One can actually prove that $\log Z(U,\theta)/\sigma$ is ergodic, in the sense that for almost all U the distribution of $\log Z(U,\theta)/\sigma$ over θ is the same as the distribution of $\log Z(U,\theta)/\sigma$ over U, as $N \to \infty$.

Theorem (CPH, Keating and O'Connell). *Denote by m the uniform probability measure on $(-\pi,\pi]$ (so that $m(\mathrm{d}\theta) = \mathrm{d}\theta/2\pi$). The sequence of laws*

$$\left\{ m \circ \left(\frac{\log Z(U,\theta)}{\sqrt{\frac{1}{2}\log N}} \right)^{-1} \right\}$$

converges weakly in probability to $X + \mathrm{i}Y$, where X, Y are iid standard normal random variables.

The moments of the zeta function

Theorem (Baker and Forrester, Keating and Snaith). *For any fixed* θ,

$$\mathbb{E}\left\{|Z(U,\theta)|^{2k}\right\} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2}$$

$$\sim \frac{G^2(k+1)}{G(2k+1)} N^{k^2} \qquad \text{as } N \to \infty$$

G is the Barnes G–function, and satisfies $G(z+1) = \Gamma(z)G(z)$.

Some results:

k	$\mathbb{E}\{\lvert Z(U,\theta)\rvert^{2k}\}$
1	N
2	$\frac{1}{12}N^4$
3	$\frac{42}{9!}N^9$
4	$\frac{24024}{16!}N^{16}$

There is a “folklore” conjecture that

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^{2k}\,\mathrm{d}t \sim g_k a(k)\left(\log\frac{T}{2\pi}\right)^{k^2}$$

where

$$a(k)=\prod_{\substack{p\\ \text{prime}}}\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}$$

k	$\frac{1}{T}\int_0^T \lvert\zeta(\frac{1}{2}+\mathrm{i}t)\rvert^{2k}\,\mathrm{d}t$
1	$\log\frac{T}{2\pi}$
2	$\frac{1}{12}a(2)(\log\frac{T}{2\pi})^4$
3	$\frac{42}{9!}a(3)(\log\frac{T}{2\pi})^9$
4	$\frac{24024}{16!}a(4)(\log\frac{T}{2\pi})^{16}$

The Keating-Snaith conjecture

Since $N = \log \frac{T}{2\pi}$, this leads to

Conjecture (Keating and Snaith).

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+\mathrm{i}t)|^{2k}\,\mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)} a(k) \left(\log\frac{T}{2\pi}\right)^{k^2}$$

as $T \to \infty$.

Very small values of $|\zeta(\frac{1}{2}+\mathrm{i}t)|$

Theorem (CPH, Keating, O'Connell). *As* $N \to \infty$, *if* $0 < y < N^{-(1/2+\epsilon)}$,

$$\mathbb{P}\left\{|Z(U,0)| \le y\right\} \sim G^2(\tfrac{1}{2}) N^{1/4} y.$$

Proof.

$$\mathbb{P}\left\{|Z(U,0)| \le y\right\} = \int_{-\infty}^{\log y} p(t)\ \mathrm{d}t$$

where

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}xt}\, \mathbb{E}\left\{ |Z(U,0)|^{\mathrm{i}x} \right\}\ \mathrm{d}x$$

When $t < -(\frac{1}{2}+\epsilon)\log N$, the Fourier integral is dominated by the pole at $x = \mathrm{i}$, and so,

$$\begin{aligned} p(t) &\sim \mathrm{i} \operatorname*{Res}_{x=\mathrm{i}} \left\{ e^{-\mathrm{i}xt}\, \mathbb{E}\left\{ |Z(U,0)|^{\mathrm{i}x} \right\} \right\} \\ &\sim e^t G^2(\tfrac{1}{2}) N^{1/4} \end{aligned}$$

Define $P(T,y)$ to equal

$$\frac{1}{T}\,\mathrm{meas}\left\{0 \le t \le T \;:\; |\zeta(\tfrac{1}{2}+\mathrm{i}t)| \le y\right\}.$$

A similar calculation using the Keating-Snaith conjecture for the moments of the zeta functions leads to:

Conjecture. If $y < (\log T)^{-(1/2+\epsilon)}$, then

$$P(T,y) \sim G^2(\tfrac{1}{2})a(-\tfrac{1}{2})y\left(\log\frac{T}{2\pi}\right)^{1/4}$$

as $T \to \infty$.

From the graph: $T = 1.52 \times 10^{19}$ and $N = 42$.

$$-\log P(T,e^{-10}) + \log \mathbb{P}\left\{\log|Z(0)| \le -10\right\} \approx 0.087$$

whereas $\log a(-\frac{1}{2}) = -0.085\ldots$

As $y \to 0+$,

$$P(T,y) \sim \frac{2y}{T} \sum_{0<\gamma_n\le T} \left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^{-1}$$

which leads to

Conjecture.

$$\frac{1}{N(T)} \sum_{0<\gamma_n\le T} \left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^{-1} \sim \pi G^2(\tfrac{1}{2}) a(-\tfrac{1}{2}) \left(\log\frac{T}{2\pi}\right)^{-3/4}$$

It is natural then to ask

Question. Can random matrix theory model

$$\frac{1}{N(T)} \sum_{0<\gamma_n\le T} \left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^{2k} \;\; ?$$

Discrete moments of the derivative

We wish to study

$$\mathbb{E}\left\{\frac{1}{N}\sum_{n=1}^{N}|Z'(U,\theta_n)|^{2k}\right\} = \mathbb{E}\left\{|Z'(U,\theta_1)|^{2k}\right\}$$

by rotational invariance of Haar measure.

Theorem (CPH, Keating, O'Connell).

$$\mathbb{E}\left\{|Z'(U,\theta_1)|^{2k}\right\} \sim \frac{G^2(k+2)}{G(2k+3)} N^{k(k+2)}$$

Conjecture.

$$\frac{1}{N(T)} \sum_{0<\gamma_n \le T} \left|\zeta'(\tfrac{1}{2} + \mathrm{i}\gamma_n)\right|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} a(k) \left(\log \frac{T}{2\pi}\right)^{k(k+2)}$$

Agrees with all known results:

Theorem (Gonek). *Under RH,*

$$\frac{1}{N(T)} \sum_{0<\gamma_n \le T} \left|\zeta'(\tfrac{1}{2} + \mathrm{i}\gamma_n)\right|^{2} \sim \frac{1}{12} \left(\log \frac{T}{2\pi}\right)^{3}$$

Conjecture (Gonek). *If all the zeros are simple and RH is true, then*

$$\frac{1}{N(T)} \sum_{0<\gamma_n \le T} \left|\zeta'(\tfrac{1}{2} + \mathrm{i}\gamma_n)\right|^{-2} \sim \frac{6}{\pi^2} \left(\log \frac{T}{2\pi}\right)^{-1}$$

Taking a theorem due to Conrey, Ghosh and Gonek, and extending it beyond the range of proven applicability, one can deduce

Conjecture.

$$\frac{1}{N(T)} \sum_{0<\gamma_n \leq T} \left|\zeta'(\tfrac{1}{2} + \mathrm{i}\gamma_n)\right|^4 \sim \frac{1}{1440\pi^2} \left(\log \frac{T}{2\pi}\right)^8$$

This is of the correct order of magnitude:

Theorem (Nathan Ng).

$$\begin{aligned} &\frac{(\sqrt{97} - \sqrt{61})^2}{30240\pi^2} \left(\log \frac{T}{2\pi}\right)^8 \\ &\qquad \leq \frac{1}{N(T)} \sum_{0<\gamma_n \leq T} \left|\zeta'(\tfrac{1}{2} + \mathrm{i}\gamma_n)\right|^4 \\ &\qquad\qquad \leq \frac{(\sqrt{97} + \sqrt{61})^2}{30240\pi^2} \left(\log \frac{T}{2\pi}\right)^8 \end{aligned}$$

Numerically, the bounds are

1.392×10^{-5} 7.036×10^{-5} 1.045×10^{-3}

Theorem (Hejhal). *Under RH and an assumption that the zeros don't get too close together,*

$$\frac{\log\left|\frac{\zeta'(1/2+\mathrm{i}\gamma_n)}{\log T}\right|}{\sqrt{\frac{1}{2}\log\log T}}$$

when averaged over all zeros between T and $2T$, behaves like a standard normal random variable.

Similarly,

Theorem (CPH, Keating, O'Connell).

$$\frac{\log|Z'(U,\theta_1)| - \log N}{\sqrt{\frac{1}{2}\log N}}$$

converges in distribution to a standard normal random variable.

Summary

- Statistically zeros of zeta behave like eigenangles of a random unitary matrix.
- One can model zeta by the characteristic polynomial of a unitary matrix.
- The model works well in the local regime (central limit theorems).
- The model must be corrected by a non-random matrix (zeta-specific) term in the global regime (moments / large deviations).