APPLICATION 3 — SIMPLE ZEROS OF $\zeta(s)$

LET

$$N_s(T) = \#\{\rho=\beta+i\gamma : \zeta(\rho)=0,\ \zeta'(\rho)\neq 0,\ 0<\gamma<T\}$$

WE BELIEVE

$$N(T) = N_0(T) = N_s(T) \qquad \forall\ T>0.$$

MONTGOMERY; $RH \Rightarrow N_s(T) > \frac{2}{3} N(T)$.

CONREY-GHOSH-GONEK:

$$RH + GLH \Rightarrow N_s(T) > {}^{19}/_{27}\, N(T)$$
$$= .703\ldots\ N(T).$$

SKETCH:

$$\Big|\sum_{0<\gamma\le T} \zeta' M_N(\tfrac{1}{2}+i\gamma)\Big|^2$$

$$\le \Bigg(\sum_{\substack{0<\gamma\le T \\ \frac{1}{2}+i\gamma \text{ SIMPLE}}} 1\Bigg)\Bigg(\sum_{0<\gamma\le T}\big|\zeta' M_N(\tfrac{1}{2}+i\gamma)\big|^2\Bigg)$$

$M_N(s) = \sum_{n\le N} \frac{b_n}{n^s}$ AND b_n IS LIKE $\mu(n)$.

$$(\tfrac{1}{2}-a)N(T) \leq \frac{1}{2\pi}\int_0^T \log|GM(a+it)|\,dt + \mathcal{E}$$

$$= \frac{1}{4\pi}\int_0^T \log\left(|GM(a+it)|^2\right)dt + \mathcal{E}$$

$$\leq \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |GM(a+it)|^2\,dt\right) + \mathcal{E}$$

$$M(s) = \sum_{n \leq T^\theta} \frac{a_n}{n^s} \qquad \text{AND}$$

a_n like $\mu(n)$.

$\rightsquigarrow$ A MEAN VALUE ESTIMATE FOR

$$\int_0^T |GM(a+it)|^2\,dt$$

LEVINSON : $\theta = \tfrac{1}{2} - \epsilon$

$\Rightarrow \quad N_0(T) > (\tfrac{1}{3} + o(1))N(T)$

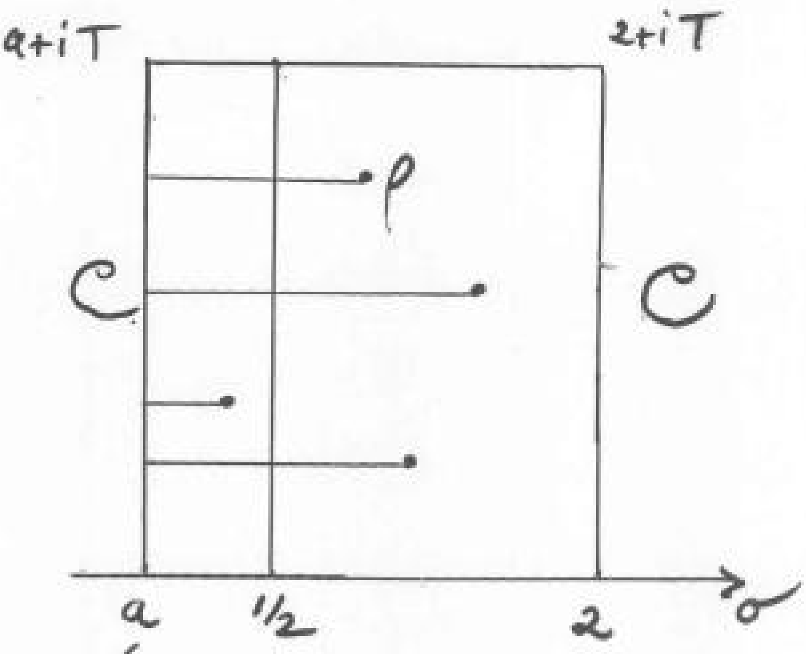
CONREY : $\theta = 4/7 - \epsilon$

$\Rightarrow \quad N_0(T) > (\tfrac{2}{5} + o(1))N(T)$

CONJECTURE (FARMER) θ ARBITRARILY LARGE WORKS.

$\Rightarrow \quad N_0(T) \sim N(T).$

ZEROS OF $G(s)$



$$a = \tfrac{1}{2} - \frac{c}{\log T}$$

LITTLEWOOD'S LEMMA :

$$\sum_{\substack{\text{ZEROS OF } G \\ \text{IN } \mathcal{C}}} \text{dist}(\rho) = \frac{1}{2\pi}\int_0^T \log \left| G(a+it) \qquad \cdot \qquad \right| dt + \mathcal{E}$$

$$\vee\!/$$

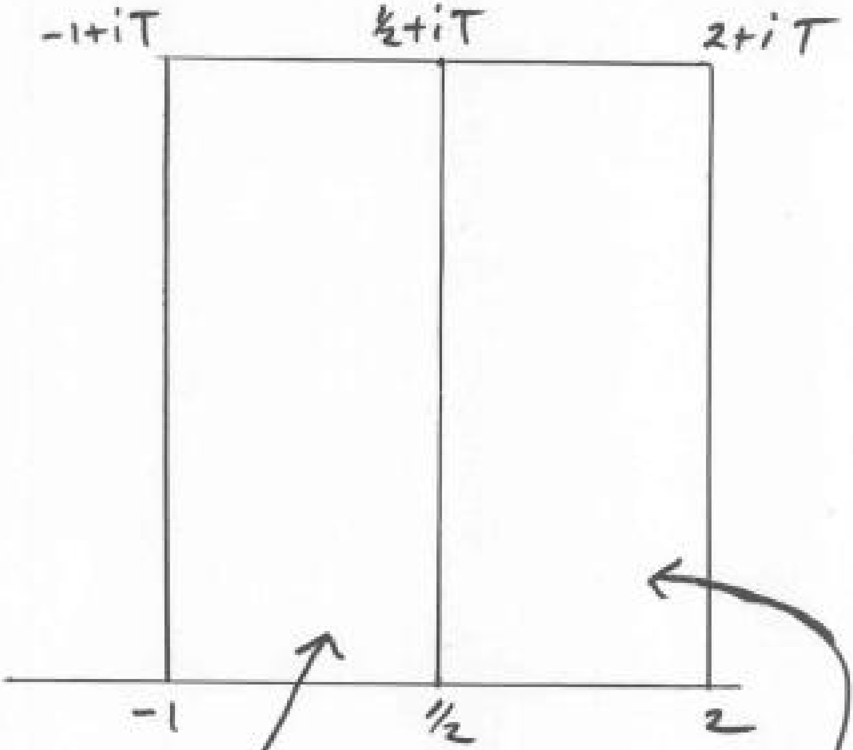
$$\sum_{\substack{\text{ZEROS OF } G \\ \text{IN } \mathcal{C} \text{ WITH } \beta > 1/2}} \text{dist}(\rho)$$

$$\vee\!/$$

$$\left(\tfrac{1}{2} - a\right) N'(T)$$

M(at; it)

M



$N'(T)$ = # ZEROS OF $\zeta'(s)$ HERE = # ZEROS OF $\zeta'(1-s)$ HERE

BY THE FUNCTIONAL EQUATION FOR $\zeta(s)$

$\zeta'(1-s)$ HAS THE SAME ZEROS AS

$$G(s) = \zeta(s) + \frac{\zeta'(s)}{L(s)}$$

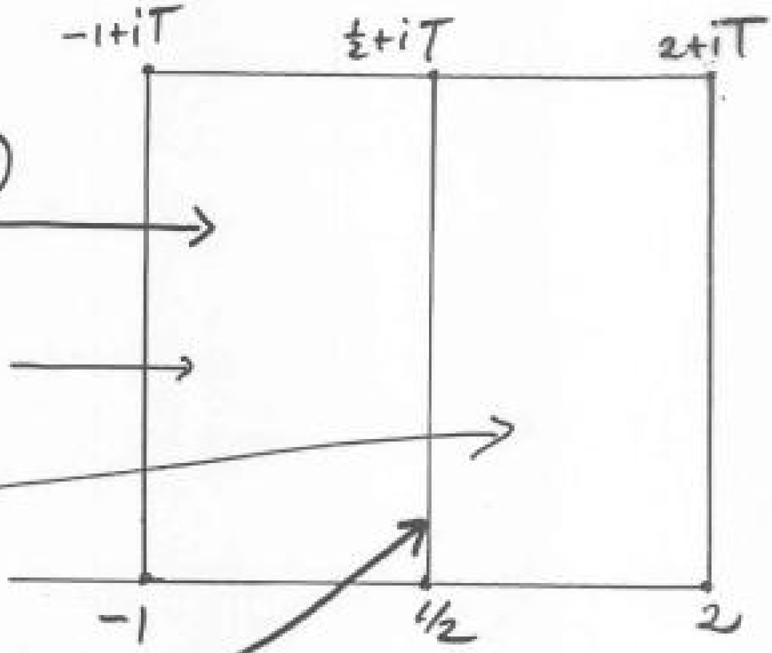
$$L(s) \approx \tfrac{1}{2} \log s .$$

SO WE NEED AN UPPER BOUND FOR THE # ZEROS OF $G(s)$ IN THE RIGHT-HAND RECTANGLE.

IF $\zeta'(s)$ HAS $N'(T)$ ZEROS HERE,

SO DOES $\zeta(s)$ (UP TO $O(\log T)$)

AND HERE



ALSO, $N_0(T)$ HERE ON $\sigma = 1/2$.

HENCE,

$$N(T) = N_0(T) + 2N'(T) + O(\log T)$$

OR

$$N_0(T) = N(T) - 2N'(T) + O(\log T).$$

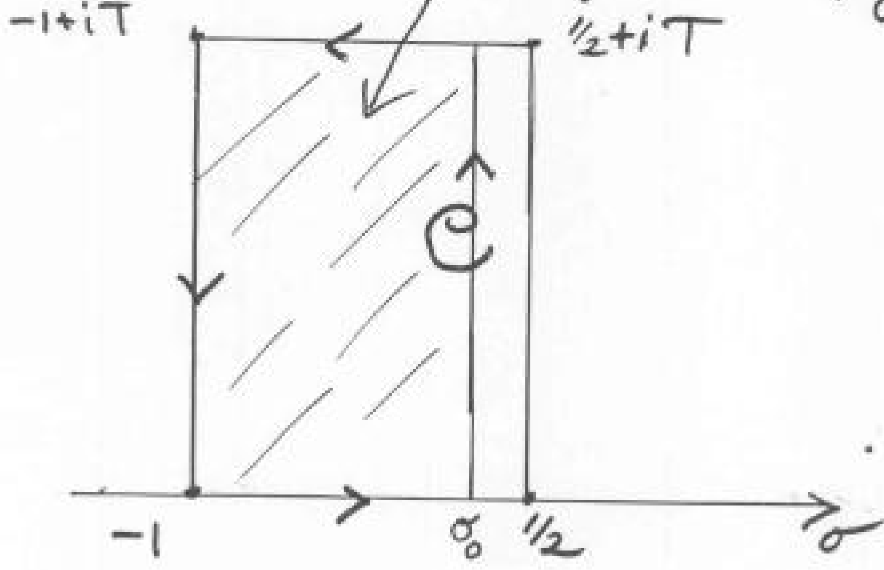
WE KNOW $N(T) \sim \frac{T}{2\pi} \log T$, SO WE NEED AN UPPER BOUND FOR $N'(T)$.

SKETCH OF THE PROOF

SPEISER: RH $\Leftrightarrow \zeta'(s) \neq 0$ IN $0 < \sigma < 1/2$.

LEVINSON–MONTGOMERY:

$\zeta(s)$ AND $\zeta'(s)$ HAVE THE SAME NUMBER OF ZEROS IN HERE (UP TO $O(\log T)$).



PROOF:

$$\Delta \arg \frac{\zeta'}{\zeta}(s)\Big|_{C} = O(\log T)$$

$$\|$$

$$2\pi\left(\#\text{ZEROS OF } \zeta'(s) - \#\text{ZEROS OF } \zeta(s)\right).$$

V. APPLICATION 2 – LEVINSON'S THEOREM

LET

$$N(T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0,\ 0 < \gamma < T\}$$
$$\sim \frac{T}{2\pi} \log T .$$

LET

$$N_0(T) = \#\{\rho = \tfrac{1}{2} + i\gamma : \zeta(\rho) = 0,\ 0 < \gamma < T\}$$

BE THE NUMBER OF ZEROS UP TO HEIGHT T ON THE CRITICAL LINE.

HARDY: $N_0(T) \to \infty$.

HARDY-LITTLEWOOD: $N_0(T) \gg T$

SELBERG: $N_0(T) \gg N(T)$

LEVINSON: $N_0(T) > \frac{1}{3} N(T)$

CONREY: $N_0(T) > \frac{2}{5} N(T)$.

TODAY WE HAVE MUCH BETTER ESTIMATES

THEY ARE OF THE FORM

$$N(\sigma, T) \ll T^{\lambda(\sigma)},$$

WHERE

$\lambda(\sigma) < 1$ AND DECREASES FOR

$$1/2 < \sigma \leq 1.$$

$$(\sigma-\sigma_0)\,N(\sigma,T) \le \frac{1}{2\pi}\int_0^T \log|\zeta(\sigma_0+it)|\,dt + \mathcal{E}$$

$$\le \frac{1}{4\pi}\int_0^T \log\left(|\zeta(\sigma_0+it)|^2\right)dt + \mathcal{E}$$

$$\le \frac{T}{4\pi}\log\Big(\underbrace{\frac{1}{T}\int_0^T |\zeta(\sigma_0+it)|^2\,dt}_{\sim\, c(\sigma_0)}\Big) + \mathcal{E}$$

$$\ll T\ .$$

THUS

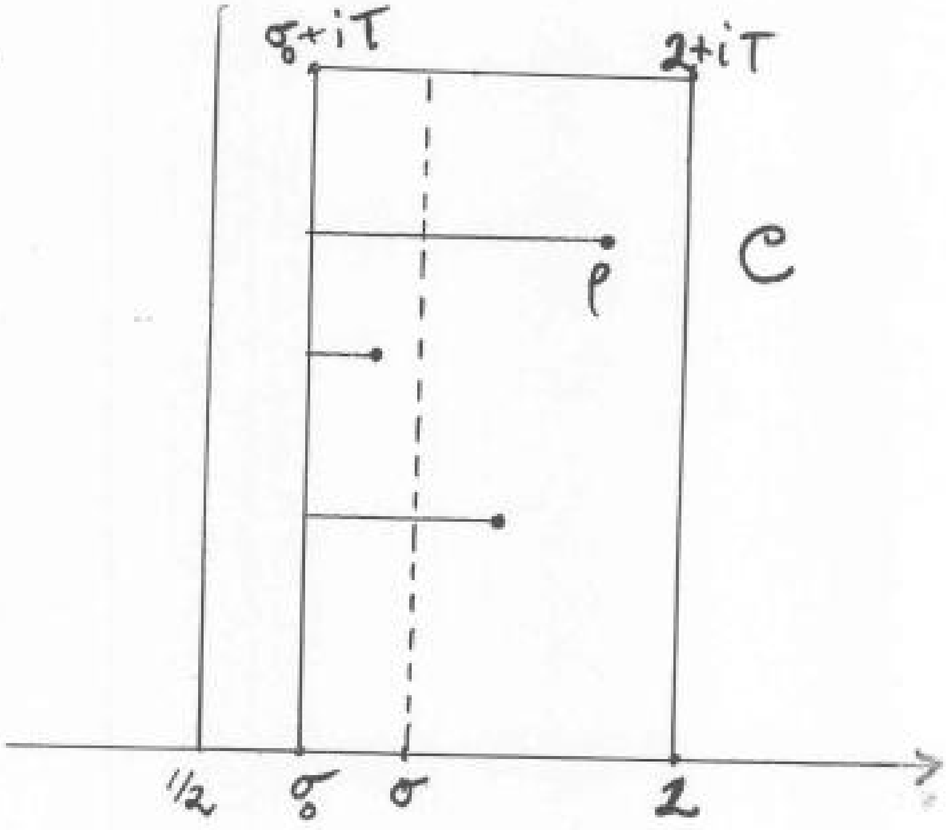
$$(\sigma-\sigma_0)\,N(\sigma,T) \ll T$$

OR

$$N(\sigma,T) \ll T$$

SINCE $N(T) \sim \frac{T}{2\pi}\log T$, ONLY AN INFINITESMAL PROPORTION OF THE ZEROS ARE OFF THE CRITICAL LINE.

THIS WAS THE FIRST ZERO-DENSITY ESTIMATE (BOHR-LANDAU (1914)).



LITTLEWOOD'S LEMMA:

$$\sum_{\rho\in\mathcal{C}} \operatorname{dist}(\rho) = \frac{1}{2\pi}\int_0^T \log\left|\zeta(\sigma_0+it)\right| dt + \mathcal{E}.$$

$$\sum_{\rho\in\mathcal{C}} \operatorname{dist}(\rho) \geq \sum_{\substack{\rho\in\mathcal{C}\\ \sigma\leq\beta}} \operatorname{dist}(\rho) \geq (\sigma-\sigma_0)N(\sigma,T).$$

HENCE

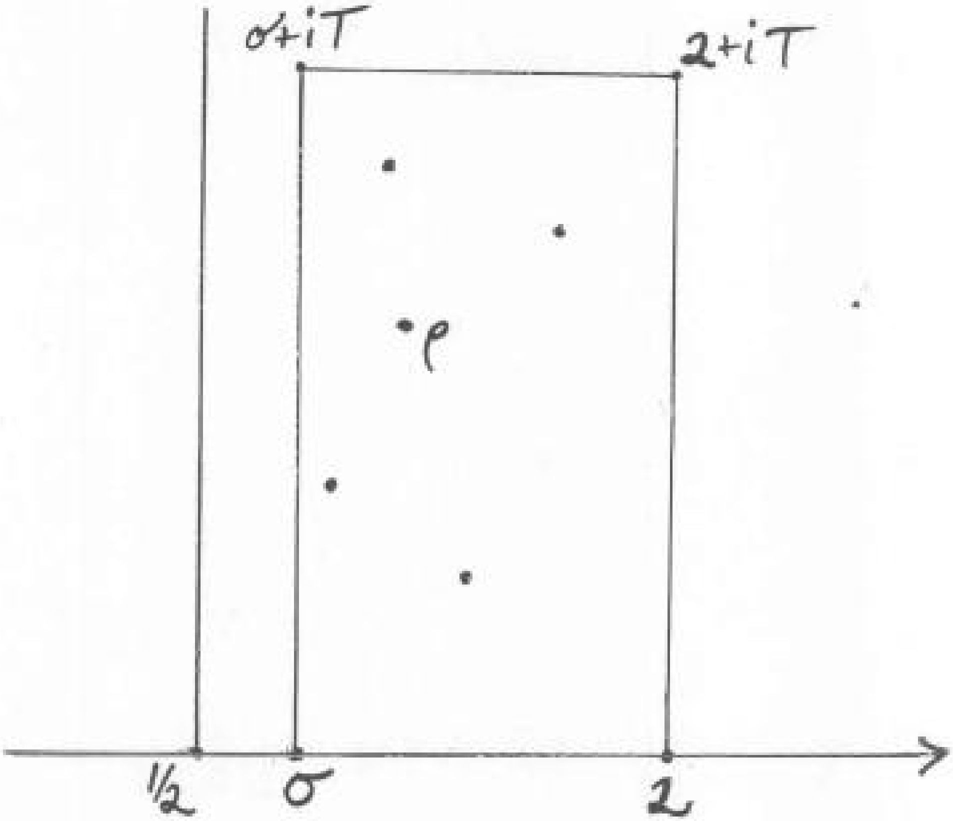
$$(\sigma-\sigma_0)N(\sigma,T) \leq \frac{1}{2\pi}\int_0^T \log\left|\zeta(\sigma_0+it)\right| dt + \mathcal{E}.$$

IV. APPLICATION 1 - A ZERO-DENSITY ESTIMATE

LET

$$N(\sigma,T) = \sum_{\substack{\rho=\beta+i\gamma \\ 0<\gamma<T \\ \sigma<\beta<1}} 1 \qquad (\sigma > 1/2).$$

WE WANT AN UPPER BOUND FOR $N(\sigma,T)$



ANOTHER IMPORTANT MEAN:

$$\int_0^T \left| \zeta^{(j)}(\sigma+it) M_N(\sigma+it) \right|^2 dt \quad ,$$

WHERE

$$M_N(s) = \sum_{n \le N} \frac{\mu(n)}{n^s} \left(1 - \frac{\log n}{\log N} \right)$$

APPROXIMATES

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (\sigma > 1)$$

NOTE: $M_N(s) \times \zeta(s)$ DAMPENS LARGE VALUES OF $\zeta(s)$ IN $\sigma > 1/2$ AND THIS CARRIES OVER TO $1/2 \le \sigma \le 1$.

MOST GENERAL ESTIMATE OF THIS TYPE IS DUE TO CONREY-GHOSH-GONEK, WITH

$$N = T^{\theta} \quad \text{AND} \quad \theta < 1/2 .$$

LATER CONREY SHOWED $\theta < 4/7$ WORKS.

NOW CONSIDER ONLY $\sigma = 1/2$.

$$I_k(\tfrac{1}{2}, T) = \int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} dt \quad .$$

<u>$k=2$:</u> INGHAM (1926):

$$I_2(\tfrac{1}{2}, T) \sim \frac{T}{2\pi^2} \log^4 T \quad .$$

<u>$k>2$:</u> NO ASYMPTOTIC YET PROVEN.

BALASUBRAMANIAN-RAMACHANDRA:

$$I_k(\tfrac{1}{2}, T) \gg T \log^{k^2} T \quad .$$

WE EXPECT

$$I_k(\tfrac{1}{2}, T) \sim c_k T \log^{k^2} T \quad .$$

<u>CONJECTURE</u> (CONREY-GHOSH):

$$c_k = \frac{a_k\, g_k}{\Gamma(k^2+1)}$$

$$a_k = \prod_p \left(\left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{r=0}^{k-1} \binom{k-1}{r}^2 p^{-r} \right)$$

$$g_k = ?$$

KEATING-SNAITH CONJECTURED g_k.

A SAMPLE OF IMPORTANT ESTIMATES

RECALL

$$I_k(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^{2k} \, dt \quad .$$

<u>$k = 1$</u> :

$$I_1(\sigma, T) \sim c(\sigma) T \qquad , \; \sigma > 1/2 .$$

HARDY - LITTLEWOOD (1918):

$$I_1(\tfrac{1}{2}, T) \sim T \log T \quad .$$

NOTE : $|\zeta(\sigma + it)|$ IS SMALLER ON AVERAGE WHEN $\sigma > 1/2$ THAN WHEN $\sigma = 1/2$.

SINCE $\zeta(\frac{1}{2} + it)$ HAS MANY ZEROS WE CAN EXPECT $\zeta(s)$ TO BE VERY ERRATIC ON $\sigma = 1/2$.

FOR US ONLY THE FIRST INTEGRAL IS SIGNIFICANT, SO WE WRITE

$$\sum_{\rho \in \mathcal{C}} \operatorname{dist}(\rho) = \frac{1}{2\pi} \int_0^T \log|f(\sigma_0 + it)| \, dt + \mathcal{E}$$

AND IGNORE $\mathcal{E}$.

$\int_0^T \log|f| \, dt$ USUALLY CANNOT BE DONE SO WE USE A TRICK:

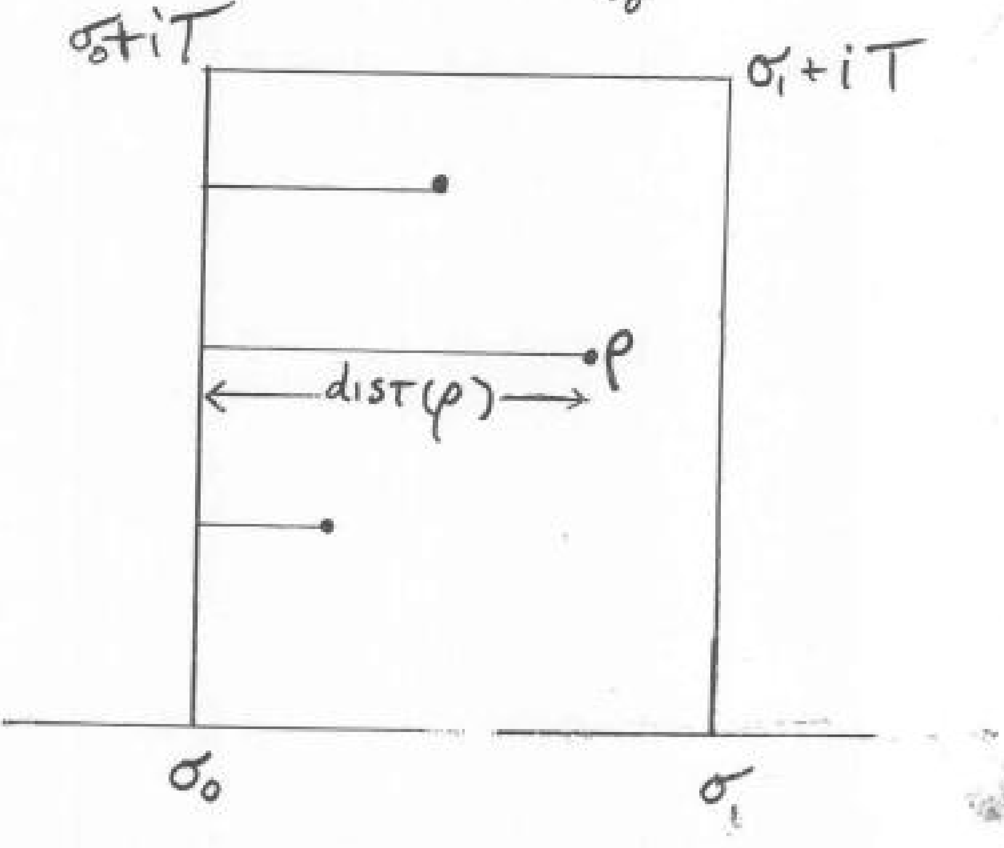
$$\frac{1}{T} \int_0^T \log|f(\sigma_0+it)| \, dt = \frac{1}{2T} \int_0^T \log\left(|f(\sigma_0+it)|^2\right) dt$$
$$\leq \frac{1}{2} \log\left(\frac{1}{T} \int_0^T |f(\sigma_0+it)|^2 \, dt\right)$$

(AVERAGE OF THE $\log$ is $\leq$ $\log$ OF THE AVERAGE.)

<u>NOTE</u>: OUR MEAN VALUES APPEAR.

LITTLEWOOD'S LEMMA LET $f(s)$ BE ANALYTIC AND NONZERO ON A RECTANGLE $\mathcal{C}$ WITH VERTICES $\sigma_0, \sigma_1, \sigma_0 + iT, \sigma_1 + iT$ WITH $\sigma_0 < \sigma_1$. LET $\operatorname{dist}(\rho)$ BE THE DISTANCE OF A ZERO ρ OF $f(s)$ FROM THE LEFT EDGE OF $\mathcal{C}$. THEN

$$2\pi \sum_{\rho \in \mathcal{C}} \operatorname{dist}(\rho) = \int_0^T \log|f(\sigma_0 + it)|\,dt - \int_0^T \log|f(\sigma_1 + it)|\,dt + \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT)\,d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\,d\sigma$$



II MEAN VALUES AND ZEROS

MEAN VALUE THEOREMS ARE USED IN MANY WAYS TO GATHER INFORMATION ON THE ZEROS. ONE DIRECT LINK IS

JENSEN'S FORMULA LET $f(z)$ BE ANALYTIC IN $|z| \leq R$ AND $f(0) \neq 0$. LET $r_1, r_2, \ldots, r_n$ BE THE MODULI OF ALL ZEROS OF $f(z)$ INSIDE $|z| \leq R$. THEN

$$\log\left(\frac{|f(0)|\,R^n}{r_1 r_2 \cdots r_n}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\,|f(Re^{i\theta})|\,d\theta$$

THE DENSITY OF ZEROS NEAR THE CENTER IS CONTROLED BY AN AVERAGE OF $f(z)$.

EXAMPLE MOMENTS OF $\zeta(s)$

$$F(s) = \zeta(s)^k \qquad \sigma_0 = 1$$

$$I_k(\sigma, T) = \int_0^T |\zeta(\sigma+it)^k|^2 \, dt$$

$$= \int_0^T |\zeta(\sigma+it)|^{2k} \, dt \, ,$$

WHERE $1/2 \le \sigma \le 1$, SAY.

EXAMPLE DIRICHLET POLYNOMIALS

$$A_N(s) = \sum_{1 \le n \le N} a_n n^{-s}$$

EASY TO SHOW THAT

$$\int_0^T |A_N(\sigma+it)|^2 \, dt = \left(T + O(N \log N)\right) \sum_{1 \le n \le N} \frac{|a_n|^2}{n^{2\sigma}}$$

MEAN VALUE THEOREMS AND THE ZEROS OF THE ZETA FUNCTION

I. WHAT IS A MEAN VALUE THEOREM?

AN ESTIMATE FOR EXPRESSIONS LIKE

$$\int_0^T F(\sigma+it)\,dt$$

OR

$$\int_0^T |F(\sigma+it)|^2\,dt$$

AS $T \to \infty$, WHERE

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad (\sigma > \sigma_1$$

BUT THE PATH OF INTEGRATION MIGHT NOT BE IN THE HALF-PLANE OF CONVERGENCE.

THERE ARE MANY VARIATIONS, LIKE

$$\sum_{r=1}^{R} |F(\sigma_r + it_r)|^2$$

FOR POINTS $s_r = \sigma_r + it_r \in \mathbb{C}$.