Multiple Zeta Functions

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(a joint work with N. Kurokawa)

Multiple Zeta Function

= A function whose zero is given by a sum of

zeros of zeta functions.

Motivation:

Possible Keys to the R.H.

[1] Additive structure of zeros
[2] Fixed width of the critical strip
(The width is always 1.)

Known Examples

Z_j: (usual) zeta function with zeros ρ_j

$Z_1 \otimes Z_2$: a multiple zeta with zeros $\rho_1 + \rho_2$

	Deligne's Theorem (Weil Conjecture)	Ramanujan Conj (Selberg e.v. conj)
Z	congruence zeta	$L(s,\varphi)$ φ: a Maass form
ρ	nontrivial zeros nontrivial poles	$s = \pm ir$ (trivial zeros)
R.H.	$\text{Re}(\rho) = \frac{k}{2}$	$\text{Re}(ir) = 0$ $\lambda = \frac{1}{4} + r^2$
critical strip	$\left\lvert \text{Re}(s) - \frac{k}{2} \right\rvert < \frac{1}{2}$	$\lvert\text{Re}(ir)\rvert < 1/2$ (Jacquet-Shalika)
$Z \otimes Z$	congruence zeta	$L\left(s, \text{Sym}^2(\varphi)\right)$
By [1]	zeros at $s = 2\rho$	$s = \pm 2ir$
R.H.	$\text{Re}(2\rho) = k$	$\text{Re}(2ir) = 0$
By [2]	$\lvert\text{Re}(2\rho) - k\rvert < \frac{1}{2}$ $\left\lvert \text{Re}(\rho) - \frac{k}{2} \right\rvert < \frac{1}{4}$	$\lvert\text{Re}(2ir)\rvert < 1/2$ $\lvert\text{Re}(ir)\rvert < 1/4$

We first calculate the simplest cases:

Multiple Hasse zeta function for finite fields.

p, q: prime numbers

$$\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1}.$$

Poles are at

$$s = 2\pi i \frac{k}{\log p} \quad (k \in \mathbb{Z}).$$

We want a new zeta function "$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$" having zeros (or poles) at

$$s = 2\pi i \left(\frac{k}{\log p} + \frac{n}{\log q} \right) \quad (k, n \in \mathbb{Z}).$$

We need some restrictions on k and n.

Theorem 1 *Let p and q be prime numbers. Define the function $\zeta(s, \mathbf{F}_p \star \mathbf{F}_q)$ as follows: for $p \neq q$*

$$\begin{aligned}
&\zeta(s, \mathbf{F}_p \star \mathbf{F}_q) \\
&\quad := \left(1 - p^{-s}\right)^{\frac{1}{2}} \left(1 - q^{-s}\right)^{\frac{1}{2}} \\
&\quad \times \exp\left(\frac{1}{2i} \sum_{k=1}^{\infty} \frac{\cot\left(\pi k \frac{\log p}{\log q}\right)}{k} p^{-ks}\right. \\
&\qquad\qquad \left. + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{\log q}{\log p}\right)}{n} q^{-ns}\right),
\end{aligned}$$

and for $p = q$

$$\begin{aligned}
&\zeta(s, \mathbf{F}_p \star \mathbf{F}_p) \\
&:= \left(1 - p^{-s}\right)^{1 - \frac{is \log p}{2\pi}} \exp\left(-\frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n^2} p^{-ns}\right).
\end{aligned}$$

Then the following properties hold:

(1) The sums over k and n converges absolutely in $\mathrm{Re}(s) > 0$.

(2) *The functions* $\zeta(s, \mathbf{F}_p \star \mathbf{F}_q)$ *are meromorphic functions of order two on the entire plane.*

(3) *The zeros of* $\zeta(s, \mathbf{F}_p \star \mathbf{F}_q)$ *are given by*

$$s = 2\pi i \left(\frac{k}{\log p} + \frac{n}{\log q} \right)$$

with k *and* n *nonnegative integers.*

(4) *The poles of* $\zeta(s, \mathbf{F}_p \star \mathbf{F}_q)$ *are given by*

$$s = 2\pi i \left(\frac{k}{\log p} + \frac{n}{\log q} \right)$$

with k *and* n *negative integers.*

Proof. The function $\zeta(s, \mathbf{F}_p \star \mathbf{F}_q)$ agrees to the absolute tensor product $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$ except for a nonzero holomorphic factor. ∎

Definition 1 (regularized product)

$$\prod_{\rho\in\mathbb{C}}(s-\rho)^{m(\rho)} := \exp\left(-\frac{\partial}{\partial w}\Big|_{w=0}\sum_{\rho\in\mathbb{C}}\frac{m(\rho)}{(s-\rho)^w}\right)$$

Definition 2 (absolute tensor product)
The absolute tensor product of zeta functions

$$Z_j(s)=\prod_{\rho\in\mathbb{C}}(s-\rho)^{m_j(\rho)} \qquad (j=1,...,r)$$

is defined by

$$(Z_1\otimes\cdots\otimes Z_r)(s) := \prod_{\rho_j\in\mathbb{C}}\Big(s-(\rho_1+\cdots+\rho_r)\Big)^{m(\rho_1,...,\rho_r)},$$

where

$$\begin{aligned} &m(\rho_1,...,\rho_r)\\ &=m(\rho_1)\cdots m(\rho_r)\times\begin{cases} 1 & \text{if } \mathrm{Im}(\rho_j)\geq 0\ (\forall j)\\ (-1)^{r-1} & \text{if } \mathrm{Im}(\rho_j)<0\ (\forall j)\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Yu. I. Manin: Lectures on zeta functions and motives (according to Deninger and Kurokawa). Asterisque 228 (1995) 121-163

Theorem 2 *Let* $\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1}$. *We have the following expressions in* $\mathrm{Re}(s) > 0$ *with* $Q(s)$ *an explicit quadratic polynomial.*

(1) When $p \neq q$,

$$\begin{aligned}
&\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \\
&\quad = e^{Q(s)} \left(1 - p^{-s}\right)^{\frac{1}{2}} \left(1 - q^{-s}\right)^{\frac{1}{2}} \\
&\qquad \times \exp\left(\frac{1}{2i} \sum_{k=1}^{\infty} \frac{\cot\left(\pi k \frac{\log p}{\log q}\right)}{k} p^{-ks} \right. \\
&\qquad\qquad \left. + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{\log q}{\log p}\right)}{n} q^{-ns} \right).
\end{aligned}$$

(2) When $p = q$,

$$\begin{aligned}
&\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p) \\
&= e^{Q(s)} \left(1 - p^{-s}\right)^{1 - \frac{is \log p}{2\pi}} \exp\left(\frac{-1}{2\pi i} \sum_{n=1}^{\infty} \frac{p^{-ns}}{n^2} \right).
\end{aligned}$$

Definition 3 (generic)
$\alpha \in \mathbb{R}$ is called *generic* if and only if

$$\lim_{m\to\infty} \|m\alpha\|^{\frac{1}{m}} = 1,$$

where $\|x\| := \min\{|x-n| \ : \ n \in \mathbb{Z}\}$ for $x \in \mathbb{R}$.

Examples 1

(1) If $\alpha \in \mathbb{Q}$, then α is not generic.

(2) If $\alpha \in (\overline{\mathbb{Q}} \cap \mathbb{R}) \setminus \mathbb{Q}$, then α is generic.

(3) Let $x, y \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$, $y \neq 1$. If $\alpha = \dfrac{\log x}{\log y} \notin \mathbb{Q}$, then α is generic. (Baker)

Baker's Theorem
Let $x, y \in \overline{\mathbb{Q}}$. For any $m, n \in \mathbb{Z}$, $m > 0$,

$$\left| m\frac{\log x}{\log y} - n \right| > m^{-c}$$

with c depending only on x and y.

Convergence

α : generic

$$\Longrightarrow \lim_{m\to\infty} \|m\alpha\|^{\frac{1}{m}} = 1$$

$$\Longrightarrow |m\alpha - n| > e^{-\varepsilon m} \quad (\forall m \geq 1, \forall n \in \mathbb{Z})$$

$$\Longrightarrow \cot(\pi m\alpha) = O(e^{\varepsilon m}) \quad (\forall \varepsilon > 0)$$

$$\Longrightarrow \sum_{m=1}^{\infty} \cot(\pi m\alpha) x^m$$

absolute convergence in $|x| < 1$

$$\alpha = \frac{\log x}{\log y}$$

$$\underset{\text{Baker}}{\Longrightarrow} |m\alpha - n| > m^{-c} \quad (\forall m \geq 1, \forall n \in \mathbb{Z})$$

$$\Longrightarrow \cot(\pi m\alpha) = O(m^c)$$

$$\Longrightarrow \sum_{m=1}^{\infty} \cot(\pi m\alpha) x^m$$

absolute convergence in $|x| < 1$

Remark 1 Assume Z_j has an analytic continuation, a functional equation and an Euler product expression in Re$(s) > \sigma_j$

$$Z_j(s) = \prod_p H_p^{(j)}(N(p)^{-s})$$

with $H_p^{(j)}(T) \in 1 + T\mathbb{C}[[T]]$. Then $Z_1 \otimes \cdots \otimes Z_r$ has an Euler product

$$\begin{aligned}&(Z_1 \otimes \cdots \otimes Z_r)(s)\\ &\quad = e^{Q(s)} \prod_{p_1,\ldots,p_r} H_{p_1,\ldots,p_r}(N(p_1)^{-s}, \ldots, N(p_r)^{-s})\end{aligned}$$

with

$$H_{p_1,\ldots,p_r}(T_1, \ldots, T_r) \in 1 + (T_1, \ldots, T_r)\mathbb{C}[[T_1, \ldots, T_r]].$$

and some polynomial $Q(s)$.

Theorem 2 means

$$\begin{aligned}&H_p^{(1)}(p^{-s}) = (1-p^{-s})^{-1},\ H_q^{(2)}(q^{-s}) = (1-q^{-s})^{-1},\\ &\qquad \Longrightarrow H_{p,q}(p^{-s}, q^{-s}) = \text{(r.h.s. of Th 2)}.\end{aligned}$$

Theorem 3 *Let* $N_0 = (N_1, N_2)$. *The following expressions hold in* $\mathrm{Re}(s) > 0$*:*

$$\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}})$$
$$= \exp\Bigg(-\frac{1}{2\pi i}\frac{N_0^2}{N_1 N_2}\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n^2} + \Big(\frac{isN_0 \log p}{2\pi} - 1\Big)\sum_{n=1}^{\infty}\frac{p^{-snN_1N_2/N_0}}{n} + \sum_{n=1}^{\infty}\frac{p^{-snN_1}f_1(n) + p^{-snN_2}f_2(n)}{n} + Q_p(s)\Bigg),$$

where $Q_p(s)$ *is a quadratic polynomial in* s *and*

$$f_1(n) = \begin{cases} (e^{2\pi i n N_1/N_2} - 1)^{-1} & (\frac{N_2}{N_0} \nmid n) \\ \frac{N_2 - N_0}{2N_0} & (\frac{N_2}{N_0} \mid n) \end{cases}$$

$$f_2(n) = \begin{cases} (e^{2\pi i n N_2/N_1} - 1)^{-1} & (\frac{N_1}{N_0} \nmid n) \\ \frac{N_1 - N_0}{2N_0} & (\frac{N_1}{N_0} \mid n) \end{cases}$$

Theorem 4 (Akatsuka (2002)) *Let p, q, r be distinct primes. In* $\mathrm{Re}(s) > 0$ *we have*

$$\begin{aligned}
&\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)\otimes\zeta(s,\mathbf{F}_r)\\
&\quad = e^{Q(s)}(1-p^{-s})^{-\frac{1}{4}}(1-q^{-s})^{-\frac{1}{4}}(1-r^{-s})^{-\frac{1}{4}}\\
&\qquad \exp\left(-\frac{1}{4}\sum_{n_1=1}^{\infty}\frac{\cot\left(\pi n_1\frac{\log p}{\log q}\right)\cot\left(\pi n_1\frac{\log p}{\log r}\right)}{n_1\ p^{n_1 s}}\right.\\
&\qquad\qquad -\frac{1}{4}\sum_{n_2=1}^{\infty}\frac{\cot\left(\pi n_2\frac{\log q}{\log p}\right)\cot\left(\pi n_2\frac{\log q}{\log r}\right)}{n_2\ q^{n_2 s}}\\
&\qquad\qquad -\frac{1}{4}\sum_{n_3=1}^{\infty}\frac{\cot\left(\pi n_3\frac{\log r}{\log p}\right)\cot\left(\pi n_3\frac{\log r}{\log q}\right)}{n_3\ r^{-n_3 s}}\\
&\qquad\qquad +\frac{i}{4}\sum_{n_1=1}^{\infty}\frac{\cot\left(\pi n_1\frac{\log p}{\log q}\right)+\cot\left(\pi n_1\frac{\log p}{\log r}\right)}{n_1\ p^{n_1 s}}\\
&\qquad\qquad +\frac{i}{4}\sum_{n_2=1}^{\infty}\frac{\cot\left(\pi n_2\frac{\log q}{\log p}\right)+\cot\left(\pi n_2\frac{\log q}{\log r}\right)}{n_2\ q^{n_2 s}}\\
&\qquad\qquad \left.+\frac{i}{4}\sum_{n_3=1}^{\infty}\frac{\cot\left(\pi n_3\frac{\log r}{\log p}\right)+\cot\left(\pi n_3\frac{\log r}{\log q}\right)}{n_3\ r^{n_3 s}}\right).
\end{aligned}$$

By putting $s = iz$,

$$\begin{aligned}\zeta(s, \mathbf{F}_p) &= (1 - p^{-s})^{-1} \\ &= p^{s/2}(p^{s/2} - p^{-s/2})^{-1} \\ &= p^{iz/2}(p^{iz/2} - p^{-iz/2})^{-1} \\ &= \frac{p^{iz/2}}{2i} \sin\left(\frac{z \log p}{2}\right)^{-1}.\end{aligned}$$

Multiple Hasse zeta function for finite fields

= Multiple sine function

The multiple Hurwitz zeta function (Barnes):

$$\zeta_r(s, z, \underline{\omega}) = \sum_{n_1,\ldots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + z)^{-s}$$

for $\underline{\omega} = (\omega_1, ..., \omega_r)$ with $\omega_j > 0$, $\mathrm{Re}(s) > r$.

The multiple gamma function:

$$\Gamma_r(z, \underline{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, z, \underline{\omega})\Big|_{s=0}\right),$$

The multiple sine function:

$$S_r(z, \underline{\omega}) = \Gamma_r(z, \underline{\omega})^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-z, \underline{\omega})^{(-1)^r}.$$

$$S_r(z) := S_r(z, (1, ..., 1))$$

$$\Gamma_r(z) := \Gamma_r(z, (1, ..., 1)).$$

$$\Gamma_1(z) = \Gamma_1(z, 1) = \Gamma(z)/\sqrt{2\pi},$$

$$S_1(z) = S_1(z, 1) = 2\sin(\pi z).$$

Proof. (of Theorem 2):

Lemma 1

$$\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)=e^{Q(s)}S_2\left(is,\left(\frac{2\pi}{\log p},\frac{2\pi}{\log q}\right)\right),$$

where $Q(s)$ is a polynomial of degree at most two, which depends on p and q.

Lemma 2 (Key Lemma) *If $\frac{\omega_1}{\omega_2}$ is generic and* $\mathrm{Im}(z)>0$,

$$\begin{aligned}
&S_2(z,(\omega_1,\omega_2))\\
&= \exp\left(\frac{1}{2i}\sum_{k=1}^{\infty}\frac{1}{k}\cot\left(\pi k\frac{\omega_2}{\omega_1}\right)e^{2\pi ik\frac{z}{\omega_1}}\right.\\
&\quad +\frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi n\frac{\omega_1}{\omega_2}\right)e^{2\pi in\frac{z}{\omega_2}}\\
&\quad +\frac{1}{2}\log\left(1-e^{2\pi i\frac{z}{\omega_1}}\right)\\
&\quad +\frac{1}{2}\log\left(1-e^{2\pi i\frac{z}{\omega_2}}\right)\\
&\quad \left.+\frac{\pi iz^2}{2\omega_1\omega_2}-\frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}+3\right)\right)
\end{aligned}$$

Proof. (of Lemma 2) We establish the "signatured" Poisson summation formula, counting only zeros in the upper half plane, with the test function

$$H(t) := (t-z)^{-2} - (t+z)^{-2}.$$

By Cauchy's theorem

$$\sum_{k,n>0} H(k\omega_1 + n\omega_2)$$
$$= \frac{1}{(2\pi i)^2} \int_C \int_C H(s_1+s_2) \frac{\xi_1'}{\xi_1}(s_1) \frac{\xi_2'}{\xi_2}(s_2) ds_1 ds_2 \tag{1}$$

with

$$\xi_1(s) = \sinh\left(\frac{\pi s}{\omega_1}\right),$$

$$\xi_2(s) = \sinh\left(\frac{\pi s}{\omega_2}\right),$$

$$C = \partial\{s \in \mathbb{C} \mid |\mathrm{Re}(s)| < \alpha,\ |s| > \alpha,\ \mathrm{Im}(s) > 0\}.$$

Computing (1) leads to

the signatured Poisson sum formula

$$\sum_{k,n>0} H(k\omega_1 + n\omega_2) \tag{2}$$

$$\begin{aligned} = \; & -\frac{1}{2}\left(\sum_{k>0} H\left(k\omega_1\right) + \sum_{n>0} H\left(n\omega_2\right)\right) \\ & -\frac{i}{2\omega_1}\sum_{k>0} \cot\left(\pi\frac{k\omega_2}{\omega_1}\right)\widetilde{H}\left(\frac{2\pi k}{\omega_1}\right) \\ & -\frac{i}{2\omega_2}\sum_{n>0} \cot\left(\pi\frac{n\omega_1}{\omega_2}\right)\widetilde{H}\left(\frac{2\pi n}{\omega_2}\right) - \frac{i}{2}\widetilde{H}'(0). \end{aligned} \tag{3}$$

On the other hand

$$(2) = \frac{d^2}{dz^2}\log S_2(z,(\omega_1,\omega_2)).$$

Thus

$$S_2(z,\omega_1,\omega_2) \;=\; \exp\left(\iint (3)\,dz\,dz\right).$$

(end of Proof of Lemma 2 and Theorem 2)

Application to special values

Theorem 5 *Let* $0 < n, k \in \mathbb{Z}$ *and put*

$$a(2n+1,k) = \sum_{l=1}^{k} (-1)^{k-l} l^{2n} \binom{2n+1}{k-l},$$

then we have

$$\begin{aligned} &\zeta(2n+1) \\ &= \frac{2^{2n+1}\pi^{2n}}{(-1)^{n-1}(2n)!} \log \prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}. \end{aligned}$$

Examples.

$$\begin{aligned} \zeta(3) &= 4\pi^2 \log S_3(1), \\ \zeta(5) &= -\frac{4\pi^4}{3} \log(S_5(1) S_5(2)^{11}), \\ \zeta(7) &= \frac{8\pi^6}{45} \log(S_7(1) S_7(2)^{57} S_7(3)^{302}). \end{aligned}$$

Theorem 6 *Let χ be a primitive odd Dirichlet character* $\pmod{N}$. *Then*

$$L(2,\chi) = \frac{2\pi i\tau(\chi)}{N^2} \log \prod_{k=1}^{N-1} \left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k \right)^{\bar{\chi}(k)}.$$

Examples.

$$\begin{aligned} &L(2,\left(\tfrac{-4}{*}\right)) \\ &= \tfrac{-\pi}{4} \log \left(S_2\left(\tfrac{1}{4}\right)^4 S_1\left(\tfrac{1}{4}\right) S_2\left(\tfrac{3}{4}\right)^{-4} S_1\left(\tfrac{3}{4}\right)^{-3} \right) \\ &= \tfrac{\pi}{4} \log \left(2^{-3} S_2\left(\tfrac{1}{4}\right)^{-8} \right), \\ &L(2,\left(\tfrac{-3}{*}\right)) \\ &= \tfrac{-2\sqrt{3}\pi}{9} \log \left(S_2\left(\tfrac{1}{3}\right)^3 S_1\left(\tfrac{1}{3}\right) S_2\left(\tfrac{2}{3}\right)^{-3} S_1\left(\tfrac{2}{3}\right)^{-2} \right) \\ &= \tfrac{4\sqrt{3}\pi}{9} \log \left(\tfrac{3}{4} S_2\left(\tfrac{1}{3}\right)^{-3} \right). \end{aligned}$$

Theorem 7 *Let* χ *be a primitive even Dirichlet character* $\pmod{N}$*.*

$$L(3,\chi) = \frac{2\pi^2\tau(\chi)}{N^3}\log\prod_{k=1}^{N-1}\left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2}\right)^{\bar{\chi}(k)}.$$

Examples.

$$L(3,\left(\tfrac{12}{*}\right))$$
$$= \frac{\sqrt{3}\pi^2}{432}\log\left(S_3\left(\frac{1}{12}\right)^{288} S_2\left(\frac{1}{12}\right)^{-408} S_1\left(\frac{1}{12}\right) S_3\left(\frac{5}{12}\right)^{-288} S_2\left(\frac{5}{12}\right)^{312} S_1\left(\frac{5}{12}\right)^{-25} S_3\left(\frac{7}{12}\right)^{-288} S_2\left(\frac{7}{12}\right)^{264} S_1\left(\frac{7}{12}\right)^{-49} S_3\left(\frac{11}{12}\right)^{288} S_2\left(\frac{11}{12}\right)^{-164} S_1\left(\frac{11}{12}\right)^{121}\right).$$

Application to Γ-factors of Selberg zeta

$M = \Gamma\backslash G/K$: a compact locally symmetric space of rank one ($\dim M$: even)

$M' = G'/K$: the compact dual symmetric space

G	K	G'	M'
$SO(1,n)$	$SO(n)$	$SO(1+n)$	S^n
$SU(1,n)$	$SU(n)$	$SU(1+n)$	$\mathbf{P}^n_{\mathbb{C}}$
$Sp(1,n)$	$Sp(n)$	$Sp(1+n)$	$\mathbf{P}^n_{\mathbf{H}}$
F_4	Spin(9)	F_4'	$\mathbf{P}^2_{\mathbf{O}}$

σ: a unitary representation of Γ

$Z_M(s,\sigma)$: The Selberg zeta function (Gangolli)

Analytic continuation to all $s \in \mathbb{C}$
as a meromorphic function of order $\dim M$.

Functional equation:

$$
\begin{aligned}
&Z_M(2\rho_0 - s, \sigma) \\
&= Z_M(s,\sigma) \exp\left(\mathrm{vol}(M) \dim(\sigma) \int_0^{s-\rho_0} \mu_M(it)dt\right)
\end{aligned}
$$

($\rho_0 > 0$, $\mu_M(t)$: the Plancherel measure)

Lemma 3 *Let $S(\Delta_{M'})$ be the set of eigenvalues of $\Delta_{M'}$. The spectral zeta function*

$$
\zeta\left(s, z, \sqrt{\Delta_{M'} + \rho_0^2}\right) := \sum_{\lambda \in S(\Delta_{M'})} \left(\sqrt{\lambda + \rho_0^2} + z\right)^{-s}
$$

is holomorphic at $s = 0$.

Thus we define

$$
\prod_{\lambda \in S(\Delta_{M'})} \left(\sqrt{\lambda + \rho_0^2} + z\right) = \det\left(\sqrt{\Delta_{M'} + \rho_0^2} + z\right).
$$

Actually

$$\det\left(\sqrt{\Delta_{M'}+\rho_0^2}+s-\rho_0\right)^{-1}$$

$$=\begin{cases}\Gamma_{2n}(s)\Gamma_{2n}(s+1) \\ \qquad\qquad\qquad\qquad (G=SO(1,2n)) \\ \prod_{k=0}^{n}\Gamma_{2n}(s+k)^{\binom{n}{k}^2} \\ \qquad\qquad\qquad\qquad (G=SU(1,n)) \\ \prod_{k=0}^{2n-1}\Gamma_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}} \\ \qquad\qquad\qquad\qquad (G=Sp(1,n)) \\ \Gamma_{16}(s)\Gamma_{16}(s+1)^{10}\Gamma_{16}(s+2)^{28} \\ \quad\times\Gamma_{16}(s+3)^{28}\Gamma_{16}(s+4)^{10}\Gamma_{16}(s+5) \\ \qquad\qquad\qquad\qquad (G=F_4)\end{cases}$$

Theorem 8 *Put*

$$\Gamma_M(s,\sigma)$$
$$=\det\left(\sqrt{\Delta_{M'}+\rho_0^2}+s-\rho_0\right)^{\mathrm{vol}(M)\dim(\sigma)(-1)^{\dim M/2}}$$

Then $\hat{Z}_M(s,\sigma)=\Gamma_M(s,\sigma)Z_M(s,\sigma)$ *satisfies the symmetric functional equation:*

$$\hat{Z}_M(s,\sigma)=\hat{Z}_M(2\rho_0-s,\sigma).$$

Proof. We prove for the case of $SO(1,2n)$.

It suffices to show

$$\exp\left(\int_0^{s-\rho_0} \mu_M(it)dt\right)^{(-1)^{\frac{\dim M}{2}}} = S_{2n}(s)S_{2n}(s+1). \tag{4}$$

Both sides are equal to 1, when $s=\rho_0=n-\frac{1}{2}$.

We compare the log derivative of (4).

We appeal to the differential equation of $S_r(z)$:

$$\frac{S_r'}{S_r}(z) = (-1)^{r-1}\binom{z-1}{r-1}\pi\cot(\pi z).$$

We also use the fact

$$\mu_M(it) = (-1)^n P_M(t)\pi\tan(\pi t),$$
$$P_M(t) = \frac{2}{(2n-1)!}t\prod_{k=1}^{n-1}\left(t^2-\left(k-\frac{1}{2}\right)^2\right).$$

Trial for the Double Riemann Zeta

$\xi(s) := \widehat{\zeta}(s + \frac{1}{2})$ with $\widehat{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

$\xi(s) = \xi(-s)$: the functional equation of $\zeta(s)$

$|\mathrm{Re}(s)| < 1/2$: the critical strip

$h(t)$: an odd function.

$H_\alpha(t) := h(2\alpha + it)$

$\widetilde{H}(u) := \int_{-\infty}^{\infty} H(t)e^{itu}dt$

Assume

$$H_\alpha(t) = O(|t|^{-2-\delta}) \qquad (|t| \to \infty)$$

for some $\delta > 0$.

Theorem 9 *(The double explicit formula)*

$$\sum_{\mathrm{Re}(\gamma_1),\mathrm{Re}(\gamma_2)>0} H_0(\gamma_1+\gamma_2)$$

$$= \frac{i}{4\pi^2}\sum_{p,q,m,n}\frac{\log p\log q}{\log(p^m q^n)}\frac{\widetilde{H_0}(\log p^{-m})-\widetilde{H_0}(\log q^{-n})}{p^m q^n}$$

$$+ \frac{i}{4\pi^2}\sum_{\substack{p,q,m,n\\ p^m\neq q^n}}\frac{\log p\log q}{\log(\frac{p^m}{q^n})}\frac{\widetilde{H_{\frac{1}{2}}}(\log p^{-m})-\widetilde{H_{\frac{1}{2}}}(\log q^{-n})}{p^m q^n}$$

$$+ \frac{1}{4\pi^2}\sum_p(\log p)^2\sum_{m=1}^{\infty}p^{-2m}\widetilde{tH_{\frac{1}{2}}(t)}(\log p^{-m})$$

$$- \frac{1}{2\pi^2}\sum_{m,p}\frac{\log p}{p^m}\int_{-\infty}^{\infty}H_{\frac{1}{2}}(t)\int_0^t\frac{\Gamma'_{\mathbb{R}}(1+it')}{\Gamma_{\mathbb{R}}(1+it')}\frac{dt'}{p^{im(t-t')}}dt$$

$$+ \frac{1}{4\pi^2}\sum_{m,p}\frac{\log p}{p^m}\int_{-\infty}^{\infty}H_0(t)\int_{-t}^t\frac{\Gamma'_{\mathbb{R}}(1+it')}{\Gamma_{\mathbb{R}}(1+it')}\frac{dt'}{p^{im(t+t')}}dt$$

$$+ \frac{1}{4\pi^2}\int_{-\infty}^{\infty}H_{\frac{1}{2}}(t)\int_0^t\frac{\Gamma'_{\mathbb{R}}(1+it_1)}{\Gamma_{\mathbb{R}}(1+it_1)}\frac{\Gamma'_{\mathbb{R}}(1+i(t-t_1))}{\Gamma_{\mathbb{R}}(1+i(t-t_1))}dt_1dt$$

$$- \frac{1}{4\pi^2}\int_{-\infty}^{\infty}H_0(t)\int_0^t\frac{\Gamma'_{\mathbb{R}}(1+it_1)}{\Gamma_{\mathbb{R}}(1+it_1)}\frac{\Gamma'_{\mathbb{R}}(1-i(t-t_1))}{\Gamma_{\mathbb{R}}(1-i(t-t_1))}dt_1dt$$

$$- \frac{1}{2\pi}\int_0^{\pi}\sum_{\mathrm{Re}(\gamma_1)>0}h(i\gamma_1+\tfrac{1}{2}e^{i\theta})\frac{\xi'}{\xi}(\tfrac{1}{2}e^{i\theta})e^{i\theta}d\theta$$

$$- \frac{1}{16\pi^2}\int_0^{\pi}\int_0^{\pi}h(\frac{e^{i\theta_1}+e^{i\theta_2}}{2})\frac{\xi'}{\xi}(\frac{e^{i\theta_1}}{2})\frac{\xi'}{\xi}(\frac{e^{i\theta_2}}{2})e^{i(\theta_1+\theta_2)}$$

$$d\theta_1 d\theta_2,$$

where the sum in the left hand side is taken over pairs $(\frac{1}{2}+i\gamma_1,\ \frac{1}{2}+i\gamma_2)$ *of nontrivial zeros of the Riemann zeta function,* p, q *denote prime numbers, and* $m, n \in \mathbb{Z}$, $m, n \geq 1$.

Ideas:

(usual cases)

Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$

$\downarrow\uparrow$ specialize $h(s)$

$$\text{Explicit formula} \quad \sum_\rho h(\rho) = \frac{1}{2\pi i}\int_C h(s)\frac{\xi'}{\xi}(s)ds. = \sum_p \text{``}\widehat{h}(p)\text{''}$$

(Multiple cases)

By Cauchy's theorem

$$\sum_{\mathrm{Im}(\rho_1),\mathrm{Im}(\rho_2)>0} h(\rho_1+\rho_2)$$
$$= \frac{1}{(2\pi i)^2}\int_{C_+}\int_{C_+} h(s_1+s_2)\frac{\xi'}{\xi}(s_1)\frac{\xi'}{\xi}(s_2)ds_1 ds_2$$
$$= \sum_{(p,q)} \text{``}\widehat{h}(p,q)\text{''}$$