Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Gaussian ensembles of random matrices I

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Introduction to Random Matr x Theory v a Gaussian Ensembles

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Consider n-dimensional Euclidean space with coordinates $(x_1, x_2, \dots, x_n)$ and the length element

$$(ds)^2 = \sum_{i=1}^{n} (dx_i)^2$$

Let a k-dimensional surface embedded in this space be parametrized by coordinates $(q_1, \dots, q_k)$, $k \le n$ as

$$x_1 = x_1(q_1, \dots, q_k), \quad \dots, \quad x_n = x_n(q_1, \dots, q_k)$$

Then the Riemannian metric $g_{ml} = g_{lm}$, $1 \le l, m \le k$ on this surface is defined via the correspondence

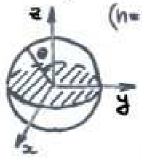
$$(ds)^2 = \sum_{i=1}^{n} (dx_i)^2 = \sum_{i=1}^{n} \Big(\sum_{m=1}^{k} \frac{\partial x_i}{\partial q_m} dq_m \Big)^2 \equiv \sum_{m,l=1}^{k} g_{ml}\, dq_m\, dq_l$$

Such a metric induces the corresponding integration measure on the surface, with the volume element

$$d\mu = \sqrt{|g|}\; dq_1 \dots dq_k\,, \quad g = \det[g_{ml}]$$

Example: Two-dimensional sphere ($k=2$) in the three-dim. ($n=3$) space: $x = R\sin\theta\cos\varphi$, $y = R\sin\theta\sin\varphi$

$z = R\cos\theta$; $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$



$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \equiv R^2 (d\theta)^2 + R^2 \sin^2\theta (d\varphi)^2$$

$$g_{11} = R^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = R^2 \sin^2\theta$$

$$g = R^4 \sin^2\theta \implies d\mu = R^2 \sin\theta\, d\theta\, d\varphi$$

Total "volume" (surface area): $\int d\mu = R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta = 4\pi R^2$

Consider a square $N \times N$ matrix $\hat{Z}$ with complex entries $z_{ij} = x_{ij} + i y_{ij}$, $1 \leq i, j \leq N$ as a point in a $2N^2$ dimensional Euclidean space with real Cartesian coordinates x_{ij}, y_{ij}.

The corresponding length element is

$$(dS)^2 = \sum_{ij} [(dx_{ij})^2 + (dy_{ij})^2] = \sum_{ij} dz_{ij} d\bar{z}_{ij} = \mathrm{Tr}(d\hat{Z}\, d\hat{Z}^*)$$

subspace of Hermitian matrices $\hat{H} = \hat{H}^*$ is specified imposing restrictions on the coordinates: $x_{ij} = x_{ji}$, $y_{ij} = -y_{ji}$

and induces the Riemannian metric:

$$(dS)^2 = \mathrm{Tr}(d\hat{H}\, d\hat{H}^*) = \sum_{i=1}^{N} (dx_{ii})^2 + 2 \sum_{i<j}^{N} [(dx_{ij})^2 + (dy_{ij})^2]$$

and the corresponding volume element:

$$d\mu(\hat{H}) = 2^{\frac{N(N-1)}{4}} \prod_{i=1}^{N} dx_{ii} \prod_{i<j} dx_{ij} dy_{ij}$$

a.k.a. the flat measure. Obviously $(dS)^2$ is invariant w.r.t. $\hat{H} \to \hat{U}^{-1} \hat{H} \hat{U}$, with $\hat{U} \in U(N)$, i.e. $\hat{U}^* \hat{U} = \hat{1}_N$

$\to d\mu(\hat{H}) = d\mu(\hat{U}^{-1} \hat{H} \hat{U})$.

The space of Hermitian matrices allows for a different coordinate system, since every $\hat{H}$ can be diagonalized:

$$\hat{H} = \hat{U} \hat{\Lambda} \hat{U}^{-1}, \quad \hat{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N), \quad \hat{U} \in U(N)$$

ensure one-to-one correspondence $\hat{H} \to (\hat{U}, \hat{\Lambda})$

we choose: $-\infty < \lambda_1 < \lambda_2 < \ldots < \lambda_N < \infty$

$$U \in U(N)/U(1) \otimes \ldots \otimes U(1) \quad (\equiv U \bmod (T^N))$$

eigenvalues λ_j are non-degenerate generically.

Problem: write down $d\mu(H)$ in the coordinates $(\hat{U}, \hat{\Lambda})$.

Sol-n: $\hat{H} = \hat{U}\hat{\Lambda}\hat{U}^* \Rightarrow d\hat{H} = \hat{U}[d\hat{\Lambda} + U^* dU \Lambda - \Lambda U^* dU]U^*$

Denote: $\delta\hat{U} \equiv \hat{U}^* d\hat{U}$ and use $\delta\hat{U}^* = -\delta\hat{U}$

The length element: $(ds)^2 = Tr(d\hat{H}\, d\hat{H}^*)$ in terms of $d\lambda_i, \delta U_{ij}$

is given by $(ds)^2 = \sum_{i=1}^{N} (d\lambda_i)^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2 \overline{\delta U_{ij}}\, \delta U_{ij}$,

introducing $\delta U_{ij} = \delta p_{ij} + i\delta q_{ij}$ we see that

$$\boxed{d\mu(H) = \prod_{i<j} (\lambda_i - \lambda_j)^2 \cdot \prod_{i=1}^{N} d\lambda_i \cdot d\mu(U)}$$

Haar's measure on $U(N)$

Now we can introduce a probability density function (p.d.f.) on the space of all Hermitian matrices $P(\hat{H})$ such that $P(\hat{H})\, d\mu(\hat{H})$ is the probability that $\hat{H}$ belongs to the volume element $d\mu(\hat{H})$.

a) Postulate of invariance: $P(\hat{H}) = P(\hat{U}^* \hat{H} \hat{U})$

$\rightarrow P(\hat{H})$ depends only on $Tr\,\hat{H}, Tr\,\hat{H}^2 \dots Tr\,\hat{H}^N$.

example: $P(\hat{H}) = C \exp\{-Tr\, Q(\hat{H})\}$

$$Q(x) = a_0 + a_2 x^2 + \dots + a_{2j} x^{2j}, \quad a_{2j} > 0$$

Observe that for $Q(x) = ax^2 + bx + c$ we have

$$e^{-Tr\,Q(\hat{H})} = e^{-cN} \cdot \prod_{i=1}^{N} e^{-ax_{ii}^2 - bx_{ii}} \cdot \prod_{i<j} e^{-2ax_{ij}^2} \cdot \prod_{i<j} e^{-2ay_{ij}^2}$$

where $\begin{array}{l} H_{ii} = x_{ii} \\ H_{ij} = x_{ij} + i y_{ij} \\ \quad i<j \end{array}$ $\Rightarrow$ variables x_{ii}, x_{ij}, y_{ij} are statistically independent

Theorem: if the p.d.f. $\mathcal{P}(\hat{H})$ is invariant w.r.t

$$\hat{H} \to \hat{H}' = \mathbf{U}^{-1} \hat{H} \mathbf{U} \text{ for all } \mathbf{U} \in \mathbf{U}(N)$$

and simultaneously all variables x_{ii}, $x_{i<j}$, $y_{i<j}$ statistically independent, then

$$\mathcal{P}(\hat{H}) = C_N \exp\{-a \,\mathrm{Tr} H^2 - b \,\mathrm{Tr} H - cN\}$$

for some real $a>0$, b, c.

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b) Information Theory approach:

Define the amount of information $I[\mathcal{P}(\hat{H})]$ associated with any probability density function $\mathcal{P}(\hat{H})$

Shanon-Khinchin entropy:

$$I[\mathcal{P}(\hat{H})] = -\int d\mu(\hat{H}) \, \mathcal{P}(\hat{H}) \ln[\mathcal{P}(\hat{H})]$$

If $\hat{H}$ is as random as possible, the corresponding $\mathcal{P}(\hat{H})$ should minimize the information, in a certain class of functions.

Natural requirements: prescribed expectation values

$$E[\mathrm{Tr}\hat{H}] \equiv \int d\mu(\hat{H}) \, \mathcal{P}(H) \,\mathrm{Tr}\hat{H} = b$$

$$E[\mathrm{Tr}\hat{H}^2] \equiv \int d\mu(\hat{H}) \, \mathcal{P}(H) \,\mathrm{Tr}\hat{H}^2 = a > 0$$

Incorporating these conditions we seek to minimize

$$\tilde{I}[\mathcal{P}(\hat{H})] = -\int d\mu(\hat{H}) \, \mathcal{P}(\hat{H}) \{\ln \mathcal{P}(\hat{H}) - \nu_1 \mathrm{Tr}\hat{H} - \nu_2 \mathrm{Tr}\hat{H}^2\}$$

Lagrange multipliers

Variation of the functional gives:

$$\delta \tilde{I}[\mathcal{P}(H)] = -\int d\mu(\hat{H})\, \delta\mathcal{P}(H) \{1 + \ln \mathcal{P}(H) - \nu_1 \mathrm{Tr}\hat{H} - \nu_2 \mathrm{Tr}\hat{H}^2\} = 0$$

possible only if $\mathcal{P}(H) \propto \exp\{\nu_1 \mathrm{Tr}\hat{H} + \nu_2 \mathrm{Tr}\hat{H}^2\}$

$\Rightarrow$ Gaussian p.d.f. minimizes the information content.

c) Brownian motion and Gaussian ensembles.

Consider a stochastic differential eq.-n. for $x(t)$:

$$\frac{d}{dt}x = -x + \xi(t), \qquad E_\xi[\xi(t_1)\xi(t_2)] = D\delta(t_1 - t_2)$$

white noise

The goal is to find p.d.f. $\mathcal{P}(t,x)$ for the variable $x(t)$ to take value x at the moment t if we know $x(0) = x_0$

Ornstein-Uhlenbeck solution:

$$\mathcal{P}(t,x) = \frac{1}{\sqrt{\pi D(1-e^{-2t})}} \exp\left\{-\frac{(x - x_0 e^{-t})^2}{D(1-e^{-2t})}\right\} \xrightarrow[t\to\infty]{} \frac{1}{\sqrt{\pi D}} e^{-\frac{x^2}{D}}$$

Consider now a Hermitian matrix with entries

$H_{ii}(t) = x_i(t)$, $H_{i<j}(t) = x_{i<j}(t) + i y_{i<j}(t)$, $t \geq 0$

and such that all independent variables perform mutually uncorrelated O-U processes. Then

$$\mathcal{P}(t,\hat{H}) = C \frac{1}{\sqrt{(1-e^{-2t})^{N^2}}} \exp\left\{-\frac{1}{D(1-e^{-2t})} \mathrm{Tr}(\hat{H} - \hat{H}_0 e^{-t})^2\right\}$$

$$\xrightarrow[t\to\infty]{} C \exp\left\{-\frac{1}{D}\mathrm{Tr}\hat{H}^2\right\}$$ – p.d.f of Gaussian Unitary Ensemble