Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Prime number theory & the Riemann zeta-function II

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Lecture 2

Open Questions

1) ∞ many "prime twins" $n, n+2$ both prime?

2) Is every even $n \geq 4$ a sum of 2 primes?
(Goldbach's Conjecture)

3) ∞ many $p = n^2+1$?

4) ∞ many $p = 2^n \ 1$
(Mersenne primes)

5) Arbitrarily long A.P.'s all prime?

6) Is there always a prime between 2 successive squares?

A selection of achievements:

1) ∞ many $p = a^2 + b^4$ (Friedlander and Iwaniec, 1998)

2) ∞ many p s.t. $p+2$ is either prime or a product of 2 primes (Chen, 1966)

3) $\exists\, n_0$ s.t. every $n \geq n_0$, n even, can be written
$$n = p + q,$$
p prime, and q either prime

or a product of 2 primes.
(Chen, 1966)

4) ∞ many n s.t. n^2+1 is either prime or a product of 2 primes (Iwaniec, 1978)

5) $c < \frac{243}{205} = 1.185$, c constant

Then $\exists\ \infty$ many n s.t.
$[n^c]$ is prime (Rivat and Wu, 2001)

$$[x] := \max\{m \in \mathbb{Z} : m \leq x\}$$

$[n^c]$ - a "polynomial of degree c"

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6) Apart for numb
of excep ions, th is always
a prime between any 2
consecutive cubes
(m,

7) $\exists n_0$ s.t. $\geq n_0$ $\exists$
$p \in [n, \quad n^{0.525}]$
(aker, Harma d Pintz,
2 01)

If $c > 0.248$ then $\exists \infty$
many consecutive primes
$p' > p$ with $' \ p \leq c \ \lg p$
(Ma 1988)

(Recall the average of
p p s ~ $\log p$)

9) $\exists$ constant $c > 0$ st $\exists$ ∞
many consecutive pr mes
$p > p$ w th

$$p \quad p \geq c \log p \, \frac{(\log\log p)(\log\log\log\log p)}{(\log\log\log p)^2}$$

(Rank n, 938)

0) Given $q \in \mathbb{N}$ and $0 \leq a < q$
w th $(a, q) = 1$, there exist
arb trar y long strings of
consecutive pr mes, al

leaving remainder a, on

v s on by q (Shiu, 2000)

9 q 0^7, a 7,777,777

ema nder a means

7,777,777

In a list of consecutive primes, one can find 10^6 (say) consecutive entries, all ending 7777,777

The Riemann Zeta-Function

$s \in \mathbb{C}$. $s = \sigma + it$.

$$n^{-s} = \exp(-s \log n),$$

$$\log n \in \mathbb{R}.$$

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad \sigma > 1$$

Absolutely convergent.

Fixed $\delta > 0$ —— uniformly convergent for $\sigma \geqslant 1 + \delta$.

$\zeta(s)$ is holomorphic for $\sigma > 1$

The Euler Product

If $\sigma > 1$ then

$$\zeta(s) = \prod_{p} (1 \quad p^{-s})^{-1}$$

Unites additive and multiplicative structures of $\mathbb{N}$

via Fundamental Theorem of Arithmetic!

$$\prod_{p \leq x} (1 \quad p^{-s})$$

$$\sigma > 1 \Rightarrow |p^{-s}| = p^{-\sigma} < 1$$

$$(- p^{-s})^{-1} = 1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots$$

$$\prod_{p \leq x} (- p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{a_x(n)}{n^s}$$

$$a_x(n) = \# \{ p_1^{e_1} \cdots p_r^{e_r} = n : p_1 < p_2 < \cdots \leq p_r \leq x \}$$

F.T.A. $a_x(n) = 0$ or 1,

$a_x(n) = 1$ <u>for</u> $n \leq x$.

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{a_x(n)}{n^s} \right| \leq \sum_{n > x} \left| \frac{1}{n^s} \right|$$

$$= \sum_{n > x} n^{-\sigma} \to 0$$

since $\sum_1^{\infty} n^{-\sigma} < \infty$.

[redacted]

$$\lim_{x \to \infty} \prod_{p \leq x} (\quad p^{-s})^{-} = \sum^{\infty} n$$

Take logs

$$\log \zeta(s) = \sum_{p} \log(1 \quad p^{-s})$$

for su table branches

D fferentiate

$$\frac{\zeta'}{\zeta}(s) \quad - \sum_{p} \sum_{k=}^{\infty} \frac{\log p}{p^{ks}}$$

justified by local uniform convergence of the result, for $\sigma > 1$

Corollary. For $\sigma > 1$ we have

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p, & n = p^k \\ 0 & \text{otherwise} \end{cases}$$

The "von-Mangoldt function"

The Theta-Function

An apparent digression!

The Poisson Summation Formula

Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and f, f', f'' all integrable on $\mathbb{R}$. Define the Fourier transform by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x} \, dx.$$

Then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

both sides converging absolutely.

$\hat{f}$ - number theorists' definition.

Proof (e.g. Rademacher)
harmonic analysis on $\mathbb{R}^+$
-additive structure only.

Take $f(x) = \exp(-x^2 \pi v)$.

$$\hat{f}(n) = \int_{-\infty}^{\infty} e^{-x^2 \pi v} e^{-2\pi i n x} \, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi v (x + in/v)^2} e^{-\pi n^2/v} \, d$$

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$$= e^{-\pi n^2/v} \int_{-\infty}^{\infty} e^{-\pi v y^2} \, dy$$

$$= \frac{1}{\sqrt{v}} e^{-\pi n^2/v}$$

So with

$$\theta(v) := \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 v)$$

Poisson implies

$$\theta(v) = \frac{1}{\sqrt{v}} \theta\left(\frac{1}{v}\right)$$

θ - a "theta-function", an example of a "modular form", since $\theta(v+2i) = \theta(v)$ too.

The "Langlands Philosophy"

"All reasonable generalizations of $\zeta(s)$ are related to modular forms, suitably generalized"

For $\zeta(s)$ use

$$\psi(v) := \sum_{n=1}^{\infty} e^{-n^2 \pi v},$$

$$\psi(v) = \{\theta(v) - 1\}/2$$

$$2\psi(v) + 1 = \frac{1}{\sqrt{v}}\{2\psi(\tfrac{1}{}$$

$$\boxed{\psi(v) = \frac{1}{\sqrt{v}}\psi\left(\frac{1}{v}\right) + \frac{1}{2\sqrt{v}} - \frac{1}{2}}$$