Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Gaussian ensembles of random matrices II

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Variation of the functional gives:

$$\delta \tilde{I}[\mathcal{P}(H)] = -\int d\mu(\hat{H})\,\delta\mathcal{P}(H)\{1+\ln\mathcal{P}(H)-\nu_1 \mathrm{Tr}\hat{H}-\nu_2\mathrm{Tr}\hat{H}^2\}=0$$

possible only if $\mathcal{P}(H)\propto \exp\{\nu_1\mathrm{Tr}\hat{H}+\nu_2\mathrm{Tr}\hat{H}^2\}$

$\Rightarrow$ Gaussian p.d.f. minimizes the information content.

c) Brownian motion and Gaussian ensembles.

Consider a stochastic differential eq.-n. for $x(t)$:

$$\frac{d}{dt}x = -x+\xi(t)\,, \qquad E_\xi[\xi(t_1)\xi(t_2)] = D\delta(t_1-t_2)$$

white noise

The goal is to find p.d.f. $\mathcal{P}(t,x)$ for the variable $x(t)$ to take value x at the moment t if we know $x(0)=x_0$

Ornstein-Uhlenbeck solution:

$$\mathcal{P}(t,x)=\frac{1}{\sqrt{\pi D(1-e^{-2t})}}\exp\left\{-\frac{(x-x_0e^{-t})^2}{D(1-e^{-2t})}\right\}\xrightarrow[t\to\infty]{}\frac{1}{\sqrt{\pi D}}e^{-\frac{x^2}{D}}$$

Consider now a Hermitian matrix with entries

$H_{ii}(t)=x_i(t)\,,\ H_{i<j}(t)=x_{i<j}(t)+iy_{i<j}(t)\,,\quad t\geq 0$

and such that all independent variables perform mutually uncorrelated O-U processes. Then

$$\mathcal{P}(t,\hat{H}) = C\frac{1}{\sqrt{(1-e^{-2t})^{N^2}}}\exp\left\{-\frac{1}{D(1-e^{-2t})}\mathrm{Tr}(\hat{H}-\hat{H}_0e^{-t})^2\right\}$$

$$\xrightarrow[t\to\infty]{} C\exp\left\{-\frac{1}{D}\mathrm{Tr}\hat{H}^2\right\}$$ – p.d.f of Gaussian Unitary Ensemble

Main goal: starting from p.d.f. $\mathcal{P}(\hat{H})$ find statistical properties of eigenvalues $\lambda_1 \quad \lambda_N$

$$\mathcal{P}(H)\, d\mu\, H) \quad C \exp\{\ \mathrm{Tr}\, Q(H)\}\, d\mu(H)$$

$$\rightarrow \underbrace{C\cdot \exp\{-\sum_{i=1}^{N} Q(\lambda_i)\}\cdot \prod_{i<j}(\lambda_i-\lambda_j)^2}_{} \prod_{i=1}^{N} d\lambda \quad d\mathcal{M}(U)$$

joint prob density function of all eigenvalues

$\mathcal{P}(\lambda_1 \quad \lambda_N)$

Note 1 Eigenvalues are highly correlated due to the Jacobian factor $\prod_{i<j}(\lambda_i-\lambda_j)^2$

Note 2 Due to invariance of $\mathcal{P}(\lambda_1 \quad \lambda_N)$ with resp. to permutations $(\lambda_1 \quad \lambda_N) \rightarrow (\lambda_{i_1}\ \lambda_{i_2}, \quad \lambda_{i_N})$ we can discegard ordering $\lambda_1 < \quad < \lambda_N$ redefining the normalization constant accordingly.

Typical questions to ask

How many eigenvalues λ can be found on average in a given nterval $[a \quad b]$? Fluctuations?

What are the statistical properties of consegutive spacing $\lambda_2 \ \lambda_1, \lambda_3 \ \lambda_2 \quad \lambda_{i+1}\ \lambda_i \quad , \lambda_N \ \lambda_{N-1}$?

What is the statistics of the largest/smallest eigenv?

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Most teresting limit of large

Characterization of spectral sequences

Let $-\infty < \lambda_1, \dots \lambda_N < \infty$ be the positions of N points on the real axis characterized by joint p.d.f

$$\mathcal{P}(\lambda_1 \dots \lambda_N)\, d\lambda_1 \dots d\lambda_N$$

of having, regardless of labelling one point $[\lambda_1, \lambda_1 + d\lambda_1]$, another in $[\lambda_2, \lambda_2 + d\lambda_2]$ ... another in $[\lambda_N, \lambda_N + d\lambda_N]$

The statistical properties of the sequence $\{\lambda_i\}$ are characterized by n-point correlation functions

$$R_n(\underbrace{\lambda_1, \dots \lambda_n}_{n}) = \frac{N!}{(N-n)!} \int \mathcal{P}(\underbrace{\lambda_1, \dots \lambda_N}_{N})\, \underbrace{d\lambda_{n+1} \dots d\lambda_N}_{N-n}$$

These are related to statistics of the number N_B of points λ_i within any set B on the real axis.

Denote $\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$ and $\rho_N(\lambda) = \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$

characteristic function

density of points

Then $N_B = \int \chi_B(\lambda) \rho_N(\lambda)\, d\lambda \equiv N_B\{\lambda_i\}$

The expectation value $\overline{N_B}$ can be found as

$$\overline{N_B} \equiv \int d\lambda_1 \dots d\lambda_N\, \mathcal{P}(\lambda_1 \dots \lambda_N)\, N_B\{\lambda\} =$$

$$= \int d\lambda\, \chi_B(\lambda) \int d\lambda_1 \dots d\lambda_N\, \mathcal{P}(\lambda_1 \dots \lambda_N) \sum \delta(\lambda - \lambda_i) =$$

$$= \int d\lambda\, \chi_B(\lambda) \cdot \underbrace{N \int d\lambda_2 \dots d\lambda_N\, \mathcal{P}(\lambda, \lambda_2, \lambda_3, \dots, \lambda_N)}_{R_1(\lambda)}$$

Thus $\boxed{\overline{N_B} = \int_B d\lambda\, R_1(\lambda)} \Rightarrow \boxed{\overline{\rho_N(\lambda)} \equiv R_1(\lambda)}$ Mean density