Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Gaussian ensembles of random matrices III

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Character zation of spectral sequences

Let $-\infty < \lambda_1, \ldots \lambda_N < \infty$ be the positions of N points on the real axis character zed by joint p d.f

$$\mathcal{P}(\lambda_1 \ldots \lambda_N)\, d\lambda_1 \ldots d\lambda_N$$

of having, regardless of labelling one point $[\lambda_1, \lambda_1 + d\lambda_1]$, another in $[\lambda_2, \lambda_2 + d\lambda_2]$ another in $[\lambda_N, \lambda_N + d\lambda_N]$

The statistical properties of the sequence $\{\lambda_i\}$ are characte-r zed by n-point correlation functions

$$R_n(\underbrace{\lambda_1, \ldots \lambda_n}_{n}) = \frac{N!}{(N-n)!} \int \mathcal{P}(\underbrace{\lambda_1, \ldots \lambda_N}_{N})\, \underbrace{d\lambda_{n+1} \ldots d\lambda_N}_{N-n}$$

These are related to statistics of the number N_B of points λ_i within any set B on the real axis.

Denote $\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$ and $\rho_N(\lambda) = \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$

characteristic function

density of points

Then $N_B = \int \chi_B(\lambda) \rho_N(\lambda)\, d\lambda \equiv N_B\{\lambda_i\}$

The expectation value $\overline{N_B}$ can be found as

$$\overline{N_B} \equiv \int d\lambda_1 \ldots d\lambda_N\, \mathcal{P}(\lambda_1 \ldots \lambda_N)\, N_B\{\lambda_i\}$$

$$= \int d\lambda\, \chi_B(\lambda) \int d\lambda_1 \ldots d\lambda_N\, \mathcal{P}(\lambda_1 \ldots \lambda_N) \sum_{i}^{N} \delta(\lambda - \lambda_i)$$

$$= \int d\lambda\, \chi_B(\lambda) \cdot \underbrace{N \int d\lambda_2 \ldots d\lambda_N\, \mathcal{P}(\lambda, \lambda_2, \lambda_3, \ldots, \lambda_N)}_{R_1(\lambda)}$$

Thus $\boxed{\overline{N_B} = \int_B d\lambda\, R_1(\lambda)} \Rightarrow \boxed{\overline{\rho_N(\lambda)} \equiv R_1(\lambda)}$ Mean density

Similarly $N_B^2 \quad \int d\lambda d\lambda' \chi_B(\lambda)\chi_B(\lambda')\sum_{i,j=1}^{N} \delta(\lambda-\lambda_i)\delta(\lambda'-\lambda_j$

$$\int d\lambda\, d\lambda' \chi_B(\lambda)\chi_B(\lambda')\left\{\delta(\lambda-\lambda')\sum_{i=1}^{N}\delta(\lambda-\lambda_i)+\sum_{i\neq j}\delta(\lambda-\lambda_i)\delta(\lambda'-\lambda_j)\right\}$$

$$N_B + \int d\lambda d\lambda' \chi_B(\lambda)\chi_B(\lambda')\sum_{i\neq j}\delta(\lambda-\lambda_i)\delta(\lambda'-\lambda_j$$

Averaging over $\mathcal{P}(\lambda_1 \quad \lambda_N)$ immediately gives

$$N_B^2 = \overline{N_B} + \int d\lambda d\lambda' \chi_B(\lambda)\chi_B(\lambda')\cdot \underbrace{N(N-1)\int d\lambda_3 \dots d\lambda_N\, \mathcal{P}(\lambda,\lambda',\lambda_3,\dots,\lambda_N}_{R_2(\lambda,\lambda')}$$

Thus $\boxed{\overline{N_B^2}=\overline{N}_B+\int_{B\times B} R_2(\lambda,\lambda')\,d\lambda d\lambda'}$, and the variance of the number of points n B is given by

$$\Sigma_2(B) = \overline{N_B^2}-\overline{N_B}^2 \quad N_B \quad \int_{B\times B} Y_2(\lambda,\lambda)\,d\lambda d\lambda$$

where we defined the so-called cluster function

$$Y_2(\lambda\ \lambda) \quad R_1(\lambda)R_1\ \lambda') \quad R_2(\lambda\ \lambda)$$

F nally one more frequently used characteristics of the spectrum is called the hole probability the probability that no points exist in the interval (L/2, L/2) It g ven by

$$A\ L) \quad \int \int d\lambda_1 \quad d\lambda_N\, \mathcal{P}(\lambda_1 \quad \lambda_N)\prod_{k=1}^{N}[1-\chi_L(\lambda_k)]$$

$$\sum_{j=0}^{N}(-1)^j\int \int d\lambda_1 \quad d\lambda_N\, \mathcal{P}(\lambda_1, \quad \lambda_N)\underbrace{h_j[\chi_L(\lambda_1),\dots,\chi_L(\lambda_N)]}_{\text{symmetric functions}}$$

Symmetric functions: $h_0(x_1,\dots,x_N)=1$,

$$h_1(x_1,\dots,x_N)=\sum_{i=1}^{N} x_i\,,\quad h_2(x_1,\dots,x_N)=\sum_{i<j} x_i x_j\,,$$

$$h_N(x_1,\dots,x_N)=x_1x_2\dots x_N.$$

We see that $h_j(\dots)$ contains $\binom{N}{j}=\frac{N!}{j!\,(N-j)!}$ terms, and

$$\int \int d\lambda_1 \;\; d\lambda_N\, \mathcal{P}(\lambda_1,\dots,\lambda_N)\, h_j[\chi_L(\lambda_1),\dots,\chi_L(\lambda_N)]$$

$$=\frac{\cancel{N!}}{j!\,\cancel{(N-j)!}}\cdot\frac{\cancel{(N-j)!}}{\cancel{N!}}\int_{-L/2}^{L/2} d\lambda_1\dots\int_{-L/2}^{L/2} d\lambda_j\; R_j(\lambda_1,\dots,\lambda_j)$$

We see that the hole **probability** $A(L)$ is expressed in terms of n-point correlation functions as

$$A(L)=\sum_{j=0}^{N}\frac{(-1)^j}{j!}\int_{-L/2}^{L/2}\cdot\int_{-L/2}^{L/2} d\lambda_1\dots d\lambda_j\; R_j(\lambda_1,\dots,\lambda_j)$$

Example: uncorrelated (poissonian) sequences

$$\mathcal{P}(\lambda_1,\dots,\lambda_N)=p(\lambda_1)\,p(\lambda_2)\dots p(\lambda_N)\,,\quad p(\lambda)\geq 0$$

$\int p(\lambda)\,d\lambda=1$, $\overline{N_B}=N\int_B p(\lambda)\,d\lambda$, $\Sigma_2(B)=\frac{N-1}{N}\cdot\overline{N_B}$

$A(L)=\left[1-\int_{-L/2}^{L/2} p(\lambda)\,d\lambda\right]^N$. In the limit $N\to\infty$

typical spacing is $\Delta=[\overline{p(0)}]^{-1}=\frac{1}{N\,p(0)}$. Let $\frac{L}{\Delta}=S<\infty$

Then for $B=(-L/2,L/2)$ we have: $\overline{N_B}=S=\Sigma_2(B)$, and

$A(L = e^{-S}$.

Method of orthogonal polynomials

van der Monde determinant:

$$\det\begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ \lambda_1 & \cdot & \cdot & \cdot & \lambda_N \\ \lambda_1^2 & \cdot & \cdot & \cdot & \lambda_N^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{N-1} & \cdot & \cdot & \cdot & \lambda_N^{N-1} \end{pmatrix} = (-1)^{\frac{N(N-1)}{2}} \prod_{i<j}^{N} (\lambda_i - \lambda_j)$$

any set of polynomials of degree k : $\underline{\pi_k(\lambda) = a_k \lambda^k + \ldots}$

we have

$$\prod_{i<j}^{N} (\lambda_i - \lambda_j) = \frac{(-1)^{\frac{N(N-1)}{2}}}{a_0 a_1 \ldots a_{N-1}} \det\begin{pmatrix} \pi_0(\lambda_1) & \cdots & \pi_0(\lambda_N) \\ \pi_1(\lambda_1) & \cdots & \pi_1(\lambda_N) \\ \pi_2(\lambda_1) & \cdots & \pi_2(\lambda_N) \\ \vdots & & \vdots \\ \pi_{N-1}(\lambda_1) & \cdots & \pi_{N-1}(\lambda_N) \end{pmatrix} \propto \det\left(\pi_{i-1}(\lambda_j)\right)_{i,j=1}^{N}$$

can rewrite the joint probability function as:

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) \propto \left[\prod_{i<j} (\lambda_i - \lambda_j) \exp\left\{-\frac{1}{2}\sum_{i=1}^{N} Q(\lambda_i)\right\}\right]^2$$

$$\propto \left[\det\left(e^{-\frac{1}{2}Q(\lambda_j)} \pi_{i-1}(\lambda_j)\right)_{i,j=1}^{N}\right]^2$$

$$= \det\begin{pmatrix} e^{-\frac{1}{2}Q(\lambda_1)}\pi_0(\lambda_1) & \cdots & e^{-\frac{1}{2}Q(\lambda_N)}\pi_0(\lambda_N) \\ \vdots & & \vdots \\ e^{-\frac{1}{2}Q(\lambda_1)}\pi_{N-1}(\lambda_1) & \cdots & e^{-\frac{1}{2}Q(\lambda_N)}\pi_{N-1}(\lambda_N) \end{pmatrix} \cdot \det\begin{pmatrix} e^{-\frac{1}{2}Q(\lambda_1)}\pi_0(\lambda_1) & \cdots & e^{-\frac{1}{2}Q(\lambda_1)}\pi_{N-1}(\lambda_1) \\ \vdots & \cdots & \vdots \\ e^{-\frac{1}{2}Q(\lambda_N)}\pi_0(\lambda_N) & \cdots & e^{-\frac{1}{2}Q(\lambda_N)}\pi_{N-1}(\lambda_N) \end{pmatrix}$$

$$= \det\Bigg(\underbrace{\sum_{j=1}^{N} e^{-\frac{1}{2}Q(\lambda_i)}\pi_{j-1}(\lambda_i) \cdot e^{-\frac{1}{2}Q(\lambda_k)}\pi_{j-1}(\lambda_k)}_{K_N(\lambda_i, \lambda_k) \text{ — "kernel"}}\Bigg)_{i,k=1}^{N}$$

Thus, $\boxed{\begin{gathered} \mathcal{P}(\lambda_1, \dots, \lambda_N) \propto \det\left(K_N(\lambda_i, \lambda_k)\right)_{i,k=1}^{N} \\ K_N(\lambda, \lambda') = \sum_{j=0}^{N-1} \varphi_j(\lambda)\varphi_j(\lambda'), \quad \varphi_{i-1}(\lambda) = e^{-\frac{1}{2}Q(\lambda)} \pi_{i-1}(\lambda) \end{gathered}}$

Suppose now that the polynomials $\pi(x)$ form an orthonormalized system w.r.t the weight $e^{Q(x)}$

$$\int dx\, e^{Q(x)} \pi(x)\pi\cdot(x) = \delta_{ij} \qquad >1 \quad \geq 1$$

Then the kernel $K_N(x,y)$ satisfies the reproducing property

$$\int dy\, K_N(x\ y\ K_N\ y,z) \quad \sum_{j,k=0}^{N-1} e^{\frac{1}{2}(Q(x)+Q(z))} \pi_j(x)\pi_k(z) \underbrace{\int dy\, e^{-Q(y)} \pi_j(y)\pi_k(y)}_{\delta_{jk}}$$

$$\sum_{j}^{N-1} e^{\frac{1}{2}(Q(x)+Q(z))} \pi_j(x)\pi_j(z \qquad K_N(x\ z) \qquad (A)$$

and also

$$q_N = \int dx\, K_N\ x,x) \quad \sum_{j=0}^{N-} \int e^{Q(x)} \pi_j(x)\pi\cdot(x)\,dx \quad N \quad (B)$$

<u>Theorem</u> (Gaudin Mehta)

Let $J_n\ \underline{x})\quad J_{ij})_{i,j=1}^{n}$ be $n \times n$ matrix with entries $J_{ij}\quad f(x_i, x_j)$ and the function $f(x,y)$ satisfies for some measure $d\mu(x$

(A) $\int f(x,y) f\ y,z)\, d\mu(y) \quad f(x,z)$, (B) $\int f(x,x)\, d\mu(x) \quad q$

Then $\int \det\ J_n(x))\, d\mu(x_n) \quad [q\ \ (n-1)] \det(J_{n-1})$

where the matr. J_{n-1} has the same functional form with $x = (x_1 \quad x_n)$ replaced by $x = x_1 \quad x_{n-1})$