Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Gaussian ensembles of random matrices IV

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Method of orthogonal polynomials

van der Monde determinant:

$$\det\begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ \lambda_1 & \cdot & \cdot & \cdot & \lambda_N \\ \lambda_1^2 & \cdot & \cdot & \cdot & \lambda_N^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{N-1} & \cdot & \cdot & \cdot & \lambda_N^{N-1} \end{pmatrix} = (-1)^{\frac{N(N-1)}{2}} \prod_{i<j}^{N} (\lambda_i - \lambda_j)$$

any set of polynomials of degree k : $\underline{\pi_k(\lambda) = a_k \lambda^k + \ldots}$

we have

$$\prod_{i<j}^{N}(\lambda_i - \lambda_j) = \frac{(-1)^{\frac{N(N-1)}{2}}}{a_0 a_1 \ldots a_{N-1}} \det\begin{pmatrix} \pi_0(\lambda_1) & \cdot\cdot\cdot & \pi_0(\lambda_N) \\ \pi_1(\lambda_1) & \cdot\cdot\cdot & \pi_1(\lambda_N) \\ \pi_2(\lambda_1) & \cdot\cdot\cdot & \pi_2(\lambda_N) \\ \vdots & \vdots & \vdots \\ \pi_{N-1}(\lambda_1) & \cdot\cdot\cdot & \pi_{N-1}(\lambda_N) \end{pmatrix} \propto \det\left(\pi_{i-1}(\lambda_j)\right)_{i,j=1}^{N}$$

can rewrite the joint probability function as

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) \propto \left[\prod_{i<j}(\lambda_i - \lambda_j) \exp\left\{-\frac{1}{2}\sum_{i=1}^{N} Q(\lambda_i)\right\}\right]^2$$

$$\propto \left[\det\left(e^{-\frac{1}{2}Q(\lambda_j)} \pi_{i-1}(\lambda_j)\right)_{i,j=1}^{N}\right]^2$$

$$= \det\begin{pmatrix} e^{-\frac{1}{2}Q(\lambda_1)}\pi_0(\lambda_1) & \cdots & e^{-\frac{1}{2}Q(\lambda_N)}\pi_0(\lambda_N) \\ \vdots & & \vdots \\ e^{-\frac{1}{2}Q(\lambda_1)}\pi_{N-1}(\lambda_1) & \cdots & e^{-\frac{1}{2}Q}\pi_{N-1}(\lambda_N) \end{pmatrix} \cdot \det\begin{pmatrix} e^{-\frac{1}{2}Q}\pi_0(\lambda_1) & \cdots & e^{-\frac{1}{2}Q(\lambda_1)}\pi_{N-1}(\lambda_1) \\ \vdots & \vdots & \vdots \\ e^{-\frac{1}{2}Q(\lambda_N)}\pi_0(\lambda_N) & \cdots & e^{-\frac{1}{2}Q(\lambda_N)}\pi_{N-1}(\lambda_N) \end{pmatrix}$$

$$= \det\Big(\underbrace{\sum_{j=1}^{N} e^{-\frac{1}{2}Q(\lambda_i)}\pi_{j-1}(\lambda_i) \cdot e^{-\frac{1}{2}Q(\lambda_k)}\pi_{j-1}(\lambda_k)}_{K_N(\lambda_i,\, \lambda_k) \text{ — "kernel"}}\Big)_{i,k=1}^{N}$$

Thus,
$$\boxed{\begin{aligned} &\mathcal{P}(\lambda_1,\dots,\lambda_N) \propto \det\left(K_N(\lambda_i,\lambda_k)\right)_{i,k=1}^{N} \\ &K_N(\lambda,\lambda') = \sum_{j=0}^{N-1} \varphi_j(\lambda)\varphi_j(\lambda')\,, \quad \varphi_{i-1}(\lambda) = e^{-\frac{1}{2}Q(\lambda)}\pi_{i-1}(\lambda) \end{aligned}}$$

Suppose now that the polynomials $\pi(x)$ form an orthonormalized system w.r.t the weight $e^{Q(x)}$

$$\int dx\, e^{Q(x)} \pi(x)\pi\cdot(x) = \delta_{ij} \qquad >1 \quad \geq 1$$

Then the kernel $K_N(x,y)$ satisfies the reproducing property

$$\int dy\, K_N(x\ y\ K_N\ y,z) \quad \sum_{j,k=0}^{N-1} e^{\frac{1}{2}(Q(x)+Q(z))}\pi_j(x)\pi_k(z) \underbrace{\int dy\, e^{-Q(y)}\pi_j(y)\pi_k(y)}_{\delta_{jk}}$$

$$\sum_{j}^{N-1} e^{\frac{1}{2}(Q(x)+Q(z))}\pi_j(x)\pi_j(z \qquad K_N(x\ z) \qquad (A)$$

and also

$$q_N = \int dx\, K_N\ x,x) \quad \sum_{j=0}^{N-} \int e^{Q(x)}\pi_j(x)\pi\cdot(x)\,dx \quad N \quad (B)$$

<u>Theorem</u> (Gaudin Mehta)

Let $J_n\ \underline{x})\quad J_{ij})_{i,j=1}^{n}$ be $n\times n$ matrix with entries $J_{ij}\quad f(x_i,x_j)$ and the function $f(x,y)$ satisfies for some measure $d\mu(x$

(A) $\int f(x,y)f\ y,z)\,d\mu(y)\quad f(x,z)$, (B) $\int f(x,x)\,d\mu(x)\quad q$

Then
$$\int \det\ J_n(x))\,d\mu(x_n) \quad [q\quad (n-1)]\det(J_{n-1})$$

where the matr. J_{n-1} has the same functional form with $x = (x_1 \quad x_n)$ replaced by $x = \ x_1 \quad x_{n-1})$

We therefore see that

$$\int d\lambda_N \mathcal{P}(\lambda_1, \dots, \lambda_N) = \int d\lambda_N \det\left[K_N(\lambda_i, \lambda_j)\right]_{i,j=1}^{N}$$

$= \left[\overset{\equiv N}{q_N} - (N-1)\right] \det\left[K_N(\lambda_i, \lambda_j)\right]_{i,j=1}^{N-1}$, and continuing this procedure one step further:

$$\int d\lambda_{N-1} \int d\lambda_N \mathcal{P}(\lambda_1, \dots, \lambda_N) = \int d\lambda_{N-1} \det\left[K_N(\lambda_i, \lambda_j)\right]_{i,j=1}^{N-1}$$

$$= \left[q_N - (N-2)\right] \det\left[K_N(\lambda_i, \lambda_j)\right]_{i,j=1}^{N-2}$$

Continuing by induction, we therefore get all n-point correlation functions of eigenvalues as

$$R_n(\lambda_1, \dots, \lambda_n) \equiv \frac{N!}{(N-n)!} \int d\lambda_{n+1} \cdots d\lambda_N \mathcal{P}(\lambda_1, \dots, \lambda_N)$$

$$= \det\left(K_N(\lambda_i, \lambda_j)\right)_{i,j=1}^{n} \equiv \det \begin{pmatrix} K_N(\lambda_1, \lambda_1) & \dots & K_N(\lambda_1, \lambda_n) \\ \vdots & \vdots & \vdots \\ K_N(\lambda_n, \lambda_1) & \dots & K_N(\lambda_n, \lambda_n) \end{pmatrix}$$

In particular,

$$\overline{\rho_N(\lambda)} = K_N(\lambda, \lambda) = \sum_{j=1}^{N-1} e^{-Q(\lambda)} \pi_{j-1}(\lambda) \pi_{j-1}(\lambda) \text{ – mean density of eigenvalues}$$

Two-point cluster function:

$$Y_2(\lambda, \lambda') = R_1(\lambda) R_1(\lambda') - R_2(\lambda, \lambda') = \left[K_N(\lambda, \lambda')\right]^2$$

The hole probability

$$A(L) = \sum_{j=0}^{N} \frac{(-1)^j}{j!} \int_{-L/2}^{L/2} d\lambda_1 \cdots \int_{-L/2}^{L/2} d\lambda_j \det \begin{pmatrix} K_N(\lambda_1, \lambda_1) & & K_N(\lambda_1, \lambda_j) \\ & & \\ K_N(\lambda_j, \lambda_1) & \dots & K_N(\lambda_j, \lambda_j) \end{pmatrix}$$

$\equiv \mathrm{Det}\left(\hat{1} - \mathcal{K}_N\right)$ "Fredholm determinant"

Hermite polynomials

orthogonality
recurrent relations
integral representations

Define polynomials $h_k(x)$ as

$$h_k(x) = (-1)^k e^{N\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-N\frac{x^2}{2}} = N^k x^k + .$$

and consider, for $k \geq \ell$, the integrals

$$\int_{-\infty}^{\infty} dx\, e^{-N\frac{x^2}{2}} h_\ell(x) h_k(x) = (-1)^k \int_{-\infty}^{\infty} dx\, h_\ell(x) \frac{d^k}{dx^k} e^{-N\frac{x^2}{2}}$$

$$= (-1)^{k+1} \int_{-\infty}^{\infty} dx\, h_\ell'(x) \frac{d^{k-1}}{dx^{k-1}} e^{-N\frac{x^2}{2}} = \ldots = (-1)^{2k} \int_{-\infty}^{\infty} dx\, e^{-N\frac{x^2}{2}} \frac{d^k}{dx^k} h_\ell(x).$$

Obviously, for $k > \ell$ it is zero, but $\frac{d^k}{dx^k} h_k(x) = k!\, N^k$.

Hence, the polynomials

$$\tilde{h}_k(x) = \frac{1}{\left[k!\, N^k \sqrt{\frac{2\pi}{N}}\right]^{1/2}} h_k(x)$$

are orthonormal w.r.t. the weight $e^{-N\frac{x^2}{2}}$

Gaussian integral identity

$$e^{-N\frac{x^2}{2}} = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{\infty} dq\, e^{-\frac{N}{2}q^2 + ixqN}$$

results in the integral representation

$$h_k(x) = (-iN)^k \sqrt{\frac{N}{2\pi}}\, e^{N\frac{x^2}{2}} \int_{-\infty}^{\infty} dq\, q^k e^{-\frac{N}{2}q^2 + ixqN}$$

Differentiating

$$\frac{d}{dx} h_k(x) = Nx\, h_k(x) - h_{k+1}(x) = Nk\, h_{k-1}(x)$$

where we used the recurrence relation for $h_k(x)$:

$$h_{k+1}(x) = (-1)^{k+1} e^{N\frac{x^2}{2}} \frac{d^k}{dx^k}\Big[\underbrace{\frac{d}{dx} e^{-N\frac{x^2}{2}}}\Big] \rightarrow \left(-Nx e^{-N\frac{x^2}{2}}\right)$$

exploiting Leibniz formula:

$$\boxed{h_{k+1}(x) = N[x h_k(x) - k h_{k-1}(x)]}$$

For orthonormal polynomials

$$\boxed{\left(\frac{k+1}{N}\right)^{\frac{1}{2}} \tilde{h}_{k+1}(x) = x\tilde{h}_k(x) - \left(\frac{k}{N}\right)^{\frac{1}{2}} \tilde{h}_{k-1}(x)}$$

multiply with $\tilde{h}_k(y)$

we can write:

subtracting
$$\begin{cases} \left(\frac{k+1}{N}\right)^{1/2} \tilde{h}_{k+1}(x)\tilde{h}_k(y) + \left(\frac{k}{N}\right)^{\frac{1}{2}} \tilde{h}_k(y)\tilde{h}_{k-1}(x) = x\tilde{h}_k(x)\tilde{h}_k(y) \\ \left(\frac{k+1}{N}\right)^{1/2} \tilde{h}_{k+1}(y)\tilde{h}_k(x) + \left(\frac{k}{N}\right)^{\frac{1}{2}} \tilde{h}_k(x) h_{k-1}(y) = y\tilde{h}_k(x)\tilde{h}_k(y) \end{cases}$$

Hence: $(x-y)\tilde{h}_k(x)\tilde{h}_k(y) = A_{k+1} - A_k$,

where $A_k = \left(\frac{k}{N}\right)^{1/2} [\tilde{h}_k(x)\tilde{h}_{k-1}(y) - \tilde{h}_{k-1}(x)\tilde{h}_k(y)]$

Summing up these expressions over k gives

$$(x-y)\sum_{k=1}^{n-1} \tilde{h}_k(x)\tilde{h}_k(y) = (A_2 + A_3 + \ldots + A_n) - (A_1 + \ldots + A_{n-1}) = A_n - A_1$$

use $A_1 = \frac{1}{\sqrt{N}}(\tilde{h}_1(x)\tilde{h}_0(y) - \tilde{h}_0(x)\tilde{h}_1(y)) = \sqrt{\frac{N}{2\pi}}(x-y) \equiv (x-y)\tilde{h}_0(x)\tilde{h}_0(x)$

and arrive at the Christoffel-Darboux formula

$$\boxed{\sum_{k=0}^{n-1} \tilde{h}_k(x)\tilde{h}_k(y) = \sqrt{\frac{n}{N}} \frac{\tilde{h}_{n-1}(y)\tilde{h}_n(x) - \tilde{h}_{n-1}(x)\tilde{h}_n(y)}{x-y}}$$

Gaussian Unitary-(Invariant) Ensemble: GUE

$$\mathcal{P}(\hat H)\,d\hat H \propto e^{-\frac{N}{2}\mathrm{Tr}\,\hat H^2}\,d\hat H \to \mathcal{P}\{\lambda_i\}\,d\lambda_1\dots d\lambda_N \propto \prod_{i<j}(\lambda_i-\lambda_j)^2\prod_{i=1}^{N} e^{-\frac{N}{2}\lambda_i^2}\,d\lambda_i$$

$Q(\lambda) = \frac{N}{2}\lambda^2 \to$ orthonormal polynomials $\pi_k(\lambda) \equiv \tilde h_k(\lambda)$

n-point correlation functions

$$R_n(\lambda_1,\dots,\lambda_n) = \det\left[K_N(\lambda_i,\lambda_j)\right]_{i,j=1}^{n}$$

where $\underline{K_N(\lambda,\lambda')} = e^{-\frac{N}{4}(\lambda^2+\lambda'^2)}\sum_{j=0}^{N-1}\tilde h_j(\lambda)\tilde h_j(\lambda')$ $\Big\}$ Christoffel-Darboux

$$\equiv \boxed{e^{-\frac{N}{4}(\lambda^2+\lambda'^2)}\,\frac{\tilde h_{N-1}(\lambda)\tilde h_N(\lambda')-\tilde h_{N-1}(\lambda)\tilde h_N(\lambda')}{\lambda-\lambda'}}$$

In particular, the mean eigenvalue density is given by

$$\overline{\rho_N(\lambda)} = K_N(\lambda,\lambda) = e^{-\frac{N}{2}\lambda^2}\cdot\left[\tilde h_N'(\lambda)\tilde h_{N-1}(\lambda)-\tilde h_N(\lambda)\tilde h_{N-1}'(\lambda)\right]$$

$$= \boxed{e^{-\frac{N}{2}\lambda^2}\cdot\frac{1}{(N-2)!\,N^{N-1}}\left[h_N^2(\lambda)-h_{N-1}(\lambda)h_{N+1}(\lambda)\right]}$$

Most interesting limit: $N\to\infty$

Plancherel-Rotach asymptotics: $h_{N+k}(\lambda)$ — ? (k — finite, N tends to ∞.)

$$h_{N+k}(x) = (-iN)^{N+k}\sqrt{\frac{N}{2\pi}}\int_{-\infty}^{\infty}dq\,q^{N+k}e^{-\frac{N}{2}(q-ix)^2}$$

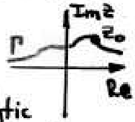
$$= (-iN)^{N+k}\sqrt{\frac{N}{2\pi}}\left[I_{N+k}(x)+(-1)^{N+k}I_{N+k}(-x)\right],\ \text{where}$$

$$I_{N+k}(x) = \int_0^{\infty}dq\cdot q^k e^{Nf(q)},\quad f(q) = \ln q-\frac{1}{2}(q-ix)^2$$

Saddle-point method of asymptotic evaluation of integrals

$$I(N) = \int_\Gamma \varphi(z)\, e^{NF(z)}\, dz$$

analytic function of z



Suppose that Γ is such that:

i) The value of $\mathrm{Re}\,F$ has its <u>maximum</u> at a point $z_0 \in \Gamma$, and decreases fast enough when we go along Γ away from z_0.

ii) The value of $\mathrm{Im}\,F$ stays <u>constant</u> along Γ,

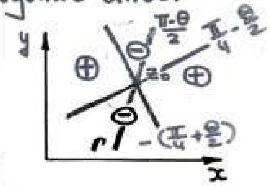
then the main contribution for large $N \gg 1$ comes from a small vicinity of $z_0 = x_0 + i y_0$.

Since $\mathrm{Re}\,F$ is a <u>harmonic</u> function of $x = \mathrm{Re}\,z$, $y = \mathrm{Im}\,z$ it can have only saddle points (x_0, y_0) found from condition of <u>stationarity</u>.

$$F'(z_0) = 0$$

In the vicinity of $z = z_0$ we can expand $F(z) \simeq F(z_0) + C(z - z_0)^2$, with $C = \frac{1}{2} F''(z_0)$. Consider level curves $\mathrm{Re}\,F(x,y) = \mathrm{Re}\,F(x_0, y_0)$ which close to z_0 are described by two orthogonal lines:

$$\begin{cases} y = y_0 + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)(x - x_0) \\ y = y_0 - \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)(x - x_0), \end{cases} \qquad C = |C| e^{i\theta}$$



partitioning (x, y) plane into four sectors: two positive $\mathrm{Re}\,F > \mathrm{Re}\,F(z_0)$, and two negative $\mathrm{Re}\,F < \mathrm{Re}\,F(z_0)$.

To apply the saddle-point method it should be possible to draw the contour Γ along the bi-sector of negative sectors: $y - y_0 = \tan\frac{\pi - \theta}{2} \cdot (x - x_0) \rightarrow \underbrace{z = z_0 + (x - x_0)\, e^{-i\frac{\pi-\theta}{2}} / \sin\frac{\theta}{2}}_{\text{local parametrization for } \Gamma}$

The leading term of large-N asymptotics:

$$\int_\Gamma \varphi(z)e^{NF(z)}dz \simeq \varphi(z_0)\sqrt{\frac{2\pi}{N|F''(z_0)|}}\,e^{NF(z_0)+\frac{i}{2}\left(\pi-\mathrm{Arg}\frac{F''(z_0)}{2}\right)}$$

Now we apply this method to our integral

$$I_{N+K}(x)=\int_0^\infty dq\, q^K e^{Nf(q)}, \quad f(q)=\ln q-\tfrac{1}{2}(q-ix)^2$$

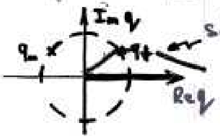
Saddle-point condition: $f'(q)=\frac{1}{q}-q+ix=0$

Two solutions: $\begin{cases} q_+=\frac{1}{2}(ix+\sqrt{4-x^2}) \\ q_-=\frac{1}{2}(ix-\sqrt{4-x^2}) \end{cases}$

Three different cases: i) $|x|<2$, ii) $|x|>2$, iii) $|x|=2$.

I) $|x|<2 \to x=2\cos\varphi, \quad 0<\varphi<\pi$

Saddle points $q_\pm=i\cos\varphi\pm\sin\varphi$, or $\begin{cases} q_+=e^{-i(\varphi-\frac{\pi}{2})} \\ q_-=e^{i(\varphi+\frac{\pi}{2})} \end{cases}$



Only q_+ is relevant, and

$\mathrm{Re}\, f(q_+)=\frac{1}{2}\cos(2\varphi), \quad \mathrm{Re}\, f(q)\big|_{q=0,\,q=\infty}\to-\infty$

hence both $q=0$ and $q=\infty$ are in negative sectors,

(in fact, in two <u>different</u> n.s.).

$f''(q_+)=-\left(1+\frac{1}{q_+^2}\right)=2i\sin\varphi\, e^{i\varphi}\to|C|=\sin\varphi, \ \theta=\varphi+\frac{\pi}{2}$

Further using $I_{N+K}(-x)=\overline{I_{N+K}(x)}$ for real x, we get

$$h_{N+K}(x)\simeq N^{N+K}\sqrt{\frac{2}{\sin\varphi}}\,e^{\frac{N}{2}\cos2\varphi}\cos\left\{(K+\tfrac{1}{2})\varphi-\tfrac{\pi}{4}+N\left(\varphi-\tfrac{1}{2}\sin2\varphi\right)\right\}$$

the required Plancherel-Rotach asymptotics