Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Pair correlation of zeros of the Riemann zeta-function and prime numbers II

D.A. Goldston (San Jose State)
31 March 2004

Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK

Tel: +44 1223 335999 Fax: +44 1223 330508
E-mail: webseminars@newton.cam.ac.uk
http://www.newton.cam.ac.uk/webseminars

Pair Correlation of Zeros and Prime Numbers

Daniel Goldston

Following Riemann, let $s \quad \sigma + it$

The Riemann zeta-function $\zeta(s)$ is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \prod_p \left(1 - \frac{1}{p^s}\right) \quad , \text{ for } \sigma > 1$$

Here p is a prime

To extract nformation about primes we compute the logarithmic derivative of $\zeta(s)$

$$
\begin{aligned}
\frac{\zeta'}{\zeta}(s) := \frac{\zeta'(s)}{\zeta(s)} \quad & \frac{d}{ds}\log\zeta(s) \\
& \frac{d}{ds}\sum_{p}\log\left(1-\frac{1}{p^s}\right) \\
& \frac{d}{ds}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{mp^{ms}} \\
& \sum_{p^m, m\geq 1}\frac{\log p}{p^{ms}} \\
& \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}, \quad \text{for} \quad \sigma > 1,
\end{aligned}
$$

where the von Mangoldt function is given by

$$
\Lambda(n) \quad \begin{cases} \log p, & \text{if } n = p^m,\ p \text{ prime},\ m > 1 \\ 0, & \text{otherwise.} \end{cases}
$$

The Chebyshev function is the counting function for $\Lambda(n)$ given by

$$
\psi(x) = \sum_{n \leq x} \Lambda(n)
$$

Because of above: $\Lambda(n)$ is preferable to indicator function for the primes

$\psi(x)$ is preferable to $\pi(x)$, the number of primes up to x.

The prime number theorem states that

$$\psi(x) \sim x, \quad \pi(x) \sim \frac{x}{\log x}, \qquad \text{as } x \to \infty,$$

The prime number theorem with the error term obtained by de la Vallée Poussin in 1899 is, for c a small constant,

$$\psi(x) = x + O(xe^{-c\sqrt{\log x}})$$

or

$$\pi(x) = \mathrm{li}(x) + O(xe^{-c\sqrt{\log x}})$$

where the logarithmic integral

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log u}\,du$$

is the actual main term in the prime number theorem.

Simple version for ar constan A 0

$$e \quad \sqrt{\log} \ll \frac{1}{\log \quad)^A}$$

where we use the Vinogradov notation $\ll$

From the Euler product for s we have

$$\zeta \; s \qquad \sum_{1}^{\infty} \frac{\mu(}{n^s} \qquad \text{for} \quad \sigma > 1$$

where the Mobius f nction μ 1 1 and

$$\mu \qquad \begin{cases} & 1 \qquad p_1 p_2 \quad p_m \; p_i \text{ s distinct} \\ 0 & \text{f } p^2 \quad \text{some } p \end{cases}$$

The zeta function has

a simple pole with residue 1 a s 1

"the trivial" zeros at s $2n$ 2 3

and complex zeros "the non trivial" zeros

$$\zeta \; \rho) \quad 0 \quad \rho \quad \beta + i\gamma \quad 0 \quad \beta < 1$$

Tha $\beta <$ is the key resu needed in the analytic proofs of the prime number theorem

The zeros are positioned symmetrically with the real line and the half line $\frac{1}{2} + it$

$$\rho \quad \rho \quad 1 \quad \rho \quad 1 \quad \rho \quad \text{are all zeros}$$

The Riemann Hypothesis RH is

$$\textbf{RH:} \quad \beta \quad \frac{}{2} \quad \text{and thus} \quad \rho \quad \frac{1}{2} + i\gamma$$

To coun complex zeros define

$$(T) \quad \gamma \quad 0 < \gamma < T$$

$$N\ T) = \frac{n(T + 0 + n(T \quad 0}{2}$$

A denote the umber of elements of A

The Riemann von Mangoldt formula for $N(T)$, obtained by applying the argument principle and using the functional equation for $\zeta(s)$, is

$$N(T) = \frac{T}{2\pi}\log(\frac{T}{2\pi e}) + 7/8 + R(T) + S(T),$$

where $R(T)$ is continuous and

$$R(T) \ll \frac{1}{T},$$

$$S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + iT) \ll \log T.$$

an estimate never improved during the last century. Thus the study of zeros is the study of $S(T)$. We conclude

$$N(T) \sim \frac{T}{2\pi}\log T.$$

Further

$$N(T+1) - N(T) = \sum_{T<\gamma\leq T+1} 1 \ll \log T,$$

a sharp estimate we use all the time

The first 5 zeros in the upper half of the critical strip are

$$\frac{1}{2} + i14.13472$$

$$\frac{1}{2} + i21.02203$$

$$\frac{1}{2} + i25.01085$$

$$\frac{1}{2} + i30.42487$$

$$\frac{1}{2} + i32.93506$$

Explicit Formulas

To do anything in this subject you need to able to work with explicit formulas.

The best known is the Riemann von Mangoldt explicit formula —"the explicit formula"
$x > 1$,

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)$$

where $\psi_0(x) = (\psi(x+0) + \psi(x-0))/2$. The sum is not absolutely convergent: ρ and $\bar{\rho}$ are grouped. The explicit formula itself contains this information: let $x \to 1^+$

Thus let $x = e^u$ $u \to 0^+$

$$\sum_\rho \frac{e^{\rho u}}{\rho} = \frac{1}{2}\log\frac{1}{u} + O(1) \quad \text{as} \quad u \to 0^+$$

applications we usually use the truncated version:

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O(\frac{x}{T}(\log xT)^2)$$
$$+ O(\log x \min\left(1, \frac{x}{T||x||}\right)),$$

last term reflecting the jumps of $\psi(x)$.

Application to PNT Assuming the Riemann Hypothesis:

$$\frac{x^\rho}{\rho} \ll \frac{x^{\frac{1}{2}}}{|\gamma|}$$

and thus

$$\sum_{|\gamma| \leq T} \frac{1}{|\gamma|} = 2 \sum_{1 < \gamma \leq T} \frac{1}{\gamma}$$
$$\leq \sum_{n=1}^{[T]+1} \frac{1}{n} \sum_{n < \gamma \leq n+1} 1$$
$$\ll \sum_{n \leq 2T} \frac{\log 2n}{n}$$
$$\ll (\log T)^2$$

Taking $T = x$ we have, assuming RH,

$$\psi(x) = x + O(x^{\frac{1}{2}}(\log x)^2),$$

proved by von Koch in 1901 and has never been improved.

This estimate implies the Riemann Hypothesis, and therefore this statement is equivalent to the Riemann Hypothesis.

Application to large gaps between primes. Let p_n denote the n$^{\text{th}}$ prime number. The highest power of a prime $\leq x$ is the largest k for which $2^k \leq x$ so $k = [\log_2 x]$. Hence by PNT

$$\psi(x) = \sum_{p \leq x} \log p + \sum_{2 \leq m \leq \log_2 x} \left(\sum_{p^m \leq x} \log p \right)$$

$$\sum_{p \leq x} \log p + \sum_{p \leq \sqrt{x}} \log p + O(\pi(x^{\frac{1}{3}})(\log x)^2)$$

$$\sum_{p \leq x} \log p + O(x^{\frac{1}{2}}),$$

nd ow c n n n t pr m pow r Thu
$1 \le h < x$

$$\sum_{<p\le\ +h} \text{og}\, p = h + O\ x^{\frac{1}{2}}\ \text{og}\, x$$

k ng h $Cx^{\frac{1}{2}}$ og x w h C r
nough con t n th u n on th ft po
v nd $\gg h$ o n v $x\ x+h$ mu t con
t n $\gg \frac{h}{\text{lo}}$ pr m f p_n th r t pr m n
$x\ x+h)$ th n

$$p\ _{+1}\ \ p\ < h\ \ x^{\frac{1}{2}}\ \text{og}\, x$$
$$\ll p_n^{\frac{1}{2}}\ \text{og}\, p_n$$

Another explici formula is the Landau formula for $> $ ixed

$$\sum_{0 < \gamma < T} x^{\rho} = -\frac{T\Lambda(x}{2\pi} + O(\log T \quad \text{as } T \to \infty$$

Here we define Λ to be zero for rea non teger This is for ixed but Gonek and separately Fujii have found usefu uniform ver sions

An explic   for n la of a least historic   teres
is the Cramér explic t formu a for Im   > 0

$$\sum_{>0} e^{\rho z} \quad \frac{e}{2\pi} \sum_{2}^{\infty} \Lambda(\quad \Big(\frac{1}{\log} + \frac{1}{\log}$$

$$+ \frac{1}{2\pi} \sum_{2} \Lambda \quad \frac{1}{+ \log} - \frac{1}{\log}$$

$$+ \frac{1}{4} + \frac{\gamma + \log 2\pi}{2\pi} \Big) \quad + ^{1}$$

$$+ \frac{1}{2\pi} \frac{\Gamma}{\Gamma} \quad 2\pi \quad + \frac{1}{2} e$$

$$2\pi i \int_0^1 e \quad \log |\zeta \quad ds$$

$$\frac{1}{2\pi i z} \int_0^{\infty} \frac{}{e^t \quad 1} \frac{dt}{+}$$

n taking $z = -\log \tau + iy$, $0 < y$ 1, and $\tau \to \infty$ Cramér proved that

$$2\pi \mathrm{Re} \sum_{\gamma > 0} e^{\rho(-\log \tau + iy)}$$
$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} \left(\frac{y}{(\log \frac{\tau}{n})^2 + y^2} \right) - \pi + O(\frac{1}{\log \tau})$$

Cramér used this formula in a series of papers starting in 1920 to prove results on primes, One result of his: Assuming RH,

$$p_{n+1} \quad p_n \ll p_n^{\frac{1}{2}} \log p_n.$$

This result only saves a logarithm over our previous almost trivial result. but this is the best result known on RH. Cramér conjectured that no gaps between consecutive primes can get very large. Recent work now suggests that Cramér's conjecture may be slightly too strong, but all evidence still suggests

$$p_{n+1} - p_n \ll (\log p_n)^2$$

There have been a number of recent papers Cramér's formula.

Most of these explic t formulas are based eva uating

$$\mathcal{I} \quad \frac{1}{2\pi} \int_{\quad \iota\infty}^{\cdot+\iota\infty} \quad \zeta \quad s \; K \quad s \quad ds$$

f equently $K \quad s \qquad K \;\; s+ \quad$ or $K \quad s \qquad K \quad zs$
If $c > 1$ then the Dirichle series converges ab solutely and

$$\mathcal{I} \qquad \sum \Lambda \qquad K$$

$$K \qquad\qquad \frac{1}{2\pi} \int_{\quad \iota\infty}^{\;+\iota\infty} K \quad s)n \quad ds$$

One then obtains the explic t formula by mov ng the contour to the left encou tering poles a s 1 and a the zeros of s as well as any poles of $K \quad s$

Another explic t form la frequently used is the Weil explic t formula or variants

The formula we wil base our work on is due to Montgomery in his origina paper on pair correlation

Proposition 1 *Assuming the Riemann Hypothesis, then for* $x \geq 1$

$$2x^{\frac{1}{2}-it}\sum_{\gamma}\frac{x^{i\gamma}}{1+(t-\gamma)^2} - \sum_{n=1}^{\infty}\frac{\Lambda(n)a_n(x)}{n^{it}} + \frac{2x^{1-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)} + x^{-\frac{1}{2}}(\log(|t|+2)+O(1)) + O(\frac{x^{-2}}{|t|+2}),$$

where

$$a_n(x) = \min\left(\left(\frac{n}{x}\right)^{\frac{1}{2}}\ \left(\frac{x}{n}\right)^{\frac{3}{2}}\right)$$

Proof. From Landau's 1909 Handbuch, (unconditionally) for $x > 1$ and $x \neq p^m$

$$\sum_{n\leq x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{x^{1-s}}{1-s} - \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{x^{-2n-s}}{2n+s}$$

provided $s \neq 1$, $s \neq \rho$, $s \neq -2n$. If $s = 0$ this is the explicit formula. Landau used this formula to prove Riemann's original explict formula for $\pi(x)$. Rewritting:

$$\sum_{\rho} \frac{x^{\rho}}{\rho - s} = -x^s\left(\frac{\zeta'}{\zeta}(s) + \sum_{n\leq x} \frac{\Lambda(n)}{n^s}\right) + \frac{x}{1-s} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n+s}$$

assume RH so $\rho = \frac{1}{2} + i\gamma$, and take s $\frac{3}{2} + it$. Then using Dirichlet series for $\frac{\zeta'}{\zeta}(s)$

$$x^{\frac{1}{2}} \sum_{\gamma} \frac{x^{i\gamma}}{1 + i(t-\gamma)} = x^{it} \sum_{n>x} \frac{\Lambda(n) a_n(x)}{n^{it}} + \frac{x}{-\frac{1}{2} - it} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n + \frac{3}{2} + it}$$

Taking next $s \quad \frac{1}{2} + it$ in original eq

$$\sum_{\gamma} \frac{i\gamma}{1 \quad i(\quad \gamma} \qquad + \sum_{n\leq} \frac{\Lambda(n) a_n}{n^{it}}$$

$$\zeta \quad \frac{1}{2} + it + \frac{3}{2} \quad it + \sum_{1} \frac{2n}{2n \quad \frac{}{2} + it}$$

Subtrac the latter from the former and use

$$\frac{\zeta'}{\zeta} \quad \frac{1}{2} + it \qquad \log \quad t| + 2 \ + O(1$$

(which follows from the unctiona equation we obtain Proposition 1 By contin ty the a ues 1 p no longer need to be exc uded

---

Montgomery's theorem

We first examine Montgomery's explicit formula heuristically: The weight in the sum over zeros concentrates the sum to zeros in a short bounded interval around t:

$$\sum_{t<\gamma\leq t+1} x^{i\gamma}$$

If this sum is substantially smaller than $\log t$ then we have detected cancelation from $x^{i\gamma}$. If x 1 or is close to 1 there can not be cancelation, and this is reflected by the term $x^{-1/2}\log(|t|+2)$ in Montgomery's formula. The sum over primes is concentrated around x:

$$\sum_{x/2<n\leq 2x} \frac{\Lambda(n)}{n^{it}}$$

The expected value of this sum is obtained by the PNT, and is exactly the remaining term

$$\frac{2x^{1-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)}$$

How does one extract information out of this formula?

Montgomery was interested in studying the distribution of differences of pairs of zeros:

Thus square the absolute value of the sum over zeros.

This would give the distribution in a bounded interval around t, but the pointwise t dependence in the Dirichlet sum over primes is intractable. Thus:

Integrate with respect to t to remove this dependence and get the distribution in a longer range. Thus work out:

$$\int_0^T \left| \sum_\gamma \frac{x^{i\gamma}}{1+(t-\gamma)^2} \right|^2 dt.$$

The weight in the sum will be small when $|\gamma - t|$ is large, which is the case over most of the integration range unless $0 < \gamma \leq T$. Thus we may restrict the sum to this range with a small error. Next, now that $0 < \gamma \leq T$ we extend the ntegration range to $(-\infty, \infty)$ with a smal error. Montgomery showed

$$\int_0^T \left| \sum_\gamma \frac{x^{i\gamma}}{1+(t-\gamma)^2} \right|^2 dt$$

$$\int_{-\infty}^{\infty} \sum_{0<\gamma\leq T} \frac{x^{i\gamma}}{1+(t-\gamma)^2}^2 dt + O((\log T)^3)$$

We multiply out the integral on the right:

$$\int_{-\infty}^{\infty} \sum_{0<\gamma\leq T} \frac{x^{i\gamma}}{1+(t-\gamma)^2}^2 dt$$

$$\frac{\pi}{2} \sum_{0<\gamma,\gamma'\leq T} x^{i(\gamma-\gamma')} w(\gamma-\gamma')$$

where

$$w(u) - \frac{4}{4+u^2}$$

this weight is obtained on evaluating the ntegral. We thus define, for $x > 0$,

$$F(x,T) = \sum_{0<\gamma,\gamma'\leq T} x^{i(\gamma-\gamma')} w(\gamma-\gamma')$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \sum_{0<\gamma\leq T} \frac{x^{i\gamma}}{1+(t-\gamma)^2}\ ^2 dt,$$

and have

$$F(x,T) \geq 0, \qquad \text{and} \qquad F(x,T) \quad F(\frac{1}{x},T),$$

and

$$F(x,T) = \frac{2}{\pi}\int_0^T \left|\sum_{\gamma} \frac{x^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt + O((\log T)^3)$$

The next step is to use Propostion 1 to evaluate $F(x,T)$. Write Montgomery's formula as

$$L(x,t) \quad R(x,t),$$

we have just shown that

$$\int_0^T \left|L(x,t)\right|^2 dt = 2\pi x F(x,T) + O(x(\log T)^3).$$

Computing the mean square of each term on for $R(x,t)$. For the Dirichlet series we use a theorem of Montgomery and Vaughan:

$$\int_0^T \Big|\sum_{n=1}^{\infty} \frac{a_n}{n^{it}}\Big|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 \big(T + o(n)\big)$$

Hence

$$\int_0^T \Big|\sum_{n=1}^{\infty} \frac{\Lambda(n) a_n(x)}{n^{it}}\Big|^2 dt = \sum_{n=1}^{\infty} (a_n(x))^2 (T + O(n))$$
$$xT(\log x + O(1)) + O(x^2 \log x),$$

using PNT with remainder. The remaining terms are:

$$\int_0^T \Big|\frac{2x^{1-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)}\Big|^2 dt \ll x^2$$

$$\int_0^T \Big|x^{-\frac{1}{2}} (\log(|t|+2) + O(1))\Big|^2 dt$$
$$\frac{T}{x}((\log T)^2 + O(\log T)),$$

and

$$\int_0^T \Big|\frac{x^{-2}}{|t|+2}\Big|^2 dt \ll x^{-4}$$

There are two main terms: the Dirichlet series term for $x \geq (\log T)^{3/2}$, and the $\log(|t| + 2)$ term for $1 \leq x \leq (\log T)^{3/4}$, and n between all terms are $o(xT \log T)$. By Cauchy-Schwartz the largest term is the main term:

$$\int_0^T |R(x,t)|^2 \, dt \quad xT(\log x + o(\log T)) + O(x^2 \log x) + \frac{T}{x}(\log T)^2(1 + o(1)).$$

We conclude

$$F(x,T) = \frac{T}{2\pi} \log x + o(T \log T) + O(x \log x) + \frac{T}{2\pi x^2}(\log T)^2(1 + o(1))$$

We tie x and T together by setting

$$x = T^{\alpha}$$

and define

$$F(\alpha) = \frac{1}{\frac{T}{2\pi} \log T} \sum_{0<\gamma,\gamma' \leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma')$$

We thus have Montgomery's theorem.

Theorem 1 *Assume the Riemann Hypothesis Then* F *is rea even and non-negative F rther for* $0 < \quad < \quad \epsilon$

$$F \ \alpha) \qquad + o(\quad + \quad + o(1 \quad T^{\ 2\alpha} \log T$$

The error term $O(\ ^2 \log$ can be mproved to $O(\ ^2$ by a sieve bound for prime twins Thus theorem holds for

$$0 < \qquad 1$$

A much more detailed analysis of the above proof has beer done by Tsz Ho Chan and all secondary terms obtained