Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Pair correlation of zeros of the Riemann zeta-function and prime numbers III

D.A. Goldston (San Jose State)

1 April 2004

Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK

Tel: +44 1223 335999   Fax: +44 1223 330508
E-mail: webseminars@newton.cam.ac.uk
http://www.newton.cam.ac.uk/webseminars

d fferences of zeros on the class of functions whose Fourier transforms are supported in 1 Using the Fourier pair

$$k(\quad \left(\frac{\sin \pi\lambda u}{\pi\lambda u}\right)^2 \quad \hat{k} \quad \lambda \max(1-|\frac{}{\lambda}|, 0$$

we have for $0 < \lambda < 1$

$$\sum_{0 \quad \gamma \quad \gamma' < T} \left(\frac{\sin(\frac{\lambda}{2}(\gamma \quad \gamma \quad \log T)}{\frac{\lambda}{2}(\gamma \quad \gamma' \log T}\right)^2 w(\gamma \quad \gamma$$

$$\left(\frac{1}{\lambda^2}\int_{-\lambda}^{\lambda} \quad \frac{|\alpha|}{\lambda} \; F \; \alpha \; d\alpha\right) \frac{T}{2\pi} \log T$$

$$\left(\frac{2}{\lambda^2}\int_0^{\lambda} 1 \quad \lambda)(\alpha + T \quad ^{2\alpha} \log T)\, d\alpha\right) \frac{T}{2\pi} \log T$$

$$\left(\frac{1}{\lambda} + \frac{\lambda}{3} \quad \frac{T}{2\pi} \log T\right.$$

From this result we easily prove

Theorem 2 *Assume the Riemann Hypothesis. At least two thirds of the zeros of the Riemann zeta-function are simple in the sense that as* $T \to \infty$

$$N_s(T) : \sum_{\substack{0<\gamma\leq T \\ \rho \text{ simple}}} 1 \geq (\frac{2}{3} - o(1)N(T)$$

Proof. The sum is over pairs of zeros counts distinct zeros weighted by their multiplicity. Thus a double pole get counted 4 times, a triple zero 9 times, etc. Letting the multiplicity of ρ be m_ρ, we have

$$\sum_{0<\gamma\leq T} m_\rho = \sum_{\substack{0<\gamma,\gamma'\leq T \\ \gamma=\gamma,}} 1$$

$$\leq \sum_{0<\gamma,\gamma'\leq T} \left(\frac{\sin(\frac{\lambda}{2}(\gamma-\gamma\ ')\log T)}{\frac{\lambda}{2}(\gamma-\gamma')\log T}\right)^2 w(\gamma-\gamma')$$

$$< (1+o(1))\left(\frac{1}{\lambda}+\frac{\lambda}{3}\right)\frac{T}{2\pi}\log T.$$

We minimize by taking $\lambda = 1$:

$$\sum_{0<\gamma\le T} m_\rho \le \Big(\frac{4}{3} + \epsilon\Big)\frac{T}{2\pi}\log T.$$

But

$$\sum_{\substack{0<\gamma\le T \\ \rho \text{ simple}}} 1 > \sum_{0<\gamma\le T} (2 - m_\rho)$$

which proves the theorem. Very sma improvements are possible here.

Conrey, Ghosh, and Gonek have proved that

$$N_s(T) \ge \left(\frac{19}{27} - \epsilon\right) N(T)$$

by a diffenent method, assuming RH and the Generealized Lindelöf Hypothesis

Montgor also proved that there are gaps between zeros closer than the average He used the Fejer pa with reversed roles to prove

$$\lim \inf_{\to\infty} \gamma + 1 \quad \gamma_n \frac{\log \gamma_n}{2\pi} < 669$$

Consider the Fourier pai

$$h \qquad \sin \pi \quad ^2 \left(\begin{matrix} 1 \\ 1 \quad u^2 \end{matrix} \right.$$

$$h \ \alpha) \quad \max(1 \qquad + \frac{\sin 2\pi|u|}{2\pi} \ 0$$

ıere h is the Selberg minorant of the characteristic function of the interva 1 1 among functions with Fourier transforms with suppor in We prove

Theorem 3 *Assuming Riemann Hypothesis*

$$\liminf_{\to\infty} \gamma_{n+1} \quad \gamma_n \frac{\log \gamma_n}{2\pi} < 6072$$

Take $r(u) = h(\frac{u}{\lambda})$ so that this a minorant of the characteristic function of the interval $[-\lambda, \lambda]$ Thus

$$\sum_{0<\gamma\leq T} m_\rho + 2 \sum_{0<\gamma-\gamma'\leq\frac{2\pi\lambda}{\log T}} 1$$

$$\geq \sum_{\gamma,\gamma'\leq T} h\Big((\gamma-\gamma')\frac{\log T}{2\pi\lambda}\Big)w(\gamma-\gamma')$$

$$\Big(\frac{T}{2\pi}\log T\Big)\int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} \lambda\hat{h}(\lambda\alpha)F(\alpha)\,d\alpha.$$

We assume that $\lambda < 1$, and since the integrand is positive we obtain a lower bound by decreasing the integration range to $[-1, 1]$ We can assume

$$\sum_{0<\gamma\leq T} m_\rho \sim \frac{T}{2\pi}\log T$$

since otherwise we have infinitely many multiple zeros and the theorem is true for that reason.

With this assumption

$$2 \sum_{0<\gamma-\gamma'\le\frac{2\pi\lambda}{\log T}} 1$$
$$\ge \frac{T}{2\pi}\log T\Big(\lambda - 1 + 2\lambda\int_0^1 \alpha h(\lambda\alpha)\,d\alpha + o(1)\Big)$$

By an easy numerical calculation we find that the right-hand side is positive for $\lambda > .6072$ which proves the result.

Better results are known by a different method. Conrey, Ghosh, and Gonek, improving a little on a result of Montgomery and Odlyzko, proved that

$$\inf_{n\to\infty}(\gamma_{n+1} - \gamma_n)\frac{\log\gamma_n}{2\pi} \le .5172.$$

Montgomery's Conjectures

What if $\alpha > 1$? Then for $x > T$

$$F(x,T)$$

$$\sim \frac{1}{2\pi x}\int_0^T \sum_{n=1}^{\infty} \frac{\Lambda(n)a_n(x)}{n^{it}} - \frac{2x^{1-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)}\Big|^2 dt$$

Now when we multiply out the off-diagnoa terms contribute and cancel the term cx from the expected value term

Assume the Hardy-Littlewood 2-tuple conjecture with a strong error term: For $0 < k \le N$,

$$\sum_{n<N} \Lambda(n)\Lambda(n+k) \quad \mathfrak{S}(k)N + O(N^{\frac{1}{2}+\epsilon})$$

where

$$\mathfrak{S}(k) \quad \begin{cases} 2C_2 \prod_{\substack{p|k \\ p>2}} \binom{p-1}{p-2} & \text{if } k \text{ is even, } n \neq 0; \\ 0, & \text{if } k \text{ is odd;} \end{cases}$$

and

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right)$$

On this basis Montgomery stated that this, if true "would allow us to carry out our program" for $x \leq T \leq x^{2-\epsilon}$ and obtain

$$F(x, T) \sim \frac{T}{2\pi} \log T$$

(For details see 1987 Diplomarbeit by Joachim Bolanz, Uber Die Montgomery'she Paarvermutung. 131 pages) (Or G-G: Long Dirichlet series . . .)

On this basis, Montgomery conjectured:

Strong Pair Correlation Conjecture For any fixed bounded number M,

$$F(\alpha) \quad 1 + o(1) \qquad \text{for } 1 < \alpha < M$$

Question What M $M(T)$ is safe here? $\log T$? T? e^T?

With this conjecture we can now evaluate any sum over differences of zeros.

Lemma 4 *Assuming RH, we have for any number B, possibly depending on T,*

$$\int_B^{B+1} F(\alpha)\,d\alpha \le 3$$

Proof With $B = C - \frac{1}{2}$,

$$\int_{C-\frac{1}{2}}^{C+\frac{1}{2}} F(\alpha)\,d\alpha$$

$$\le 2\int_{C-1}^{C+1} \Big(1 - |\alpha - C|\Big) F(\alpha)\,d\alpha$$

$$\frac{2}{\frac{T}{2\pi}\log T} \sum_{0<\gamma,\gamma'\le T} T^{iC(\gamma-\gamma')} \left(\frac{\sin(\frac{1}{2}(\gamma-\gamma')\log T)}{\frac{1}{2}(\gamma-\gamma')\log T}\right)^2 \times w(\gamma-\gamma')$$

$$\le \frac{2}{\frac{T}{2\pi}\log T} \sum_{0<\gamma,\gamma'\le T} \left(\frac{\sin(\frac{1}{2}(\gamma-\gamma')\log T)}{\frac{1}{2}(\gamma-\gamma')\log T}\right)^2 w(\gamma-\gamma')$$

$$\le \frac{8}{3} + \epsilon \quad \le 3.$$

The Pair Correlation Conjecture For fixed $\beta > 0$,

$$N(T,\beta) := \frac{1}{\frac{T}{2\pi}\log T} \sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma'-\gamma\leq\frac{2\pi\beta}{\log T}}} 1$$

$$\sim \int_0^\beta 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du$$

This mmediately shows:

$$\liminf_{n\to\infty}(\gamma_{n+1} - \gamma_n)\frac{\log \gamma_n}{2\pi} = 0$$

Almost All Simple Zeros Conjecture (SZC)
We have

$$N^*(T) := \frac{1}{\frac{T}{2\pi}\log T} \sum_{0<\gamma\leq T} m_\rho \sim 1.$$

Theorem 5 *Assuming the Riemann Hypothesis the Strong PCC for F implies the PCC and SZC conjectures*

Proof SZC As before we ind tha for λ 1

$$\sum_{0 \quad \gamma \gamma' < T} \left(\frac{\sin(\frac{\lambda}{2}(\gamma \quad \gamma \quad \log T)}{\frac{\lambda}{2} \gamma \quad \gamma' \log T} \right)^2 w(\gamma \quad \gamma$$

$$1 + \frac{}{3\lambda^2} \frac{T}{2\pi} \log T$$

Now let $\lambda \quad \infty$

PCC Using the Fejer kernel from before

$$\frac{1}{\frac{T}{2\pi} \log T} \sum_{\substack{0 < \gamma \gamma' < T \\ \gamma | < \frac{2\pi\beta}{\log T}}} \frac{1}{\beta} \left(\quad \frac{\gamma \quad \gamma' \log T}{2\pi\beta} \quad w \gamma \quad \gamma \right.$$

$$\int_{-}^{\infty} F \quad \frac{\sin(\pi\beta\alpha}{\pi\beta\alpha} \quad ^2 d\alpha$$

The left hand side is

$$N \quad T) + \frac{2}{\beta} \int_0^{\beta} N \; T \quad du + O \; \frac{\beta(1 + \beta}{\log T)^2}$$

ast error from removing $w(\gamma \quad \gamma$ F rst term we just proved is 1 We use Montgomery s Theorem for the RHS n range $|\alpha| < 1$ Strong PCC in range $1 < |\alpha| \quad M$ and the lemma for $|\alpha| > M$ getting

$$\int_0^\beta N \; T \quad du \quad \frac{\beta^2}{2} + \int_0^\beta \quad \beta \quad \frac{\sin \pi}{\pi} \quad du$$

Use

$$\frac{1}{h}\int_{\beta \;\; h}^{\beta} N \; \boldsymbol{T} \quad du \quad N \; \boldsymbol{T} \; \beta < \frac{}{\boldsymbol{h}}\int_{\beta \;\; h}^{\beta} N \; \boldsymbol{T} \quad du$$

and difference to get resu

Gallagher and Mueller on PC Conjecture

Theorem 6 *Suppose*

$$N(T,\beta) \sim \int_0^\beta 1 - \mu(\alpha)\, d\alpha,$$

uniformly for $0 \le \beta_0 \le \beta \le \beta_1 < \infty$, *where* μ *is a real, even, continuous,* L^1 *function. Then*

$$\int_{-\infty}^{\infty} \mu(\alpha)\, d\alpha \sim N^*(T)$$

and $h = \frac{2\pi\beta}{\log T}$, $\beta > 0$

$$R(T,h) := \int_0^T (S(t+h) \quad S(t))^2\, dt$$
$$\sim T \int_{-\infty}^{\infty} \min(|\alpha|, \beta)\mu(\alpha)\, d\alpha$$

Since

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = 1$$

we see that PCC implies SZC. Also

$$\int_{-\infty}^{\infty} \mu(\alpha)\, d\alpha \ge 1$$

These are proved unconditionally without suming RH

Main tool: Fujii-Selberg result:

$R(T,h) \ll T \log(2 + h \log T)$ if $\frac{1}{\log T} \ll h \ll 1$,

and

$R(T,h) \sim \frac{T}{\pi^2} \log(h \log T)$ if $h \log T \to \infty, h \ll 1$

Now

$$\int_0^T (N(t+h) - N(t))^2 dt = \int_0^T \Big(\sum_{t<\gamma\leq t+h} 1 \Big)^2$$

$$\sim \frac{T}{2\pi} \qquad \Big(N^*(T) + \frac{2}{h} \int_0^h N(T,u)\, du \Big)$$

But left side is also by Riemann von Mangoldt

$$\sim T(h \frac{\log T}{2\pi})^2 + R(T,h)$$

Subtitute for $N(T,u)$ to get $\mu(\alpha)$ into this and let $h \log T \to \infty$ and $h \to 0$ and apply Fujii–Selberg.

On RH and assuming SZC they proved

$$\mu(\quad 1 \quad |\alpha| \qquad |\alpha| < 1$$

PCC agrees with this and says tha 0
elsewhere

Heath Brown s results on Primes

On RH and assuming SZC they proved

$$\widehat{\mu}(\alpha) = 1 - |\alpha|, \qquad |\alpha| \leq 1.$$

PCC agrees with this and says that $\widehat{\mu}(\alpha) = 0$ elsewhere.

Heath-Brown's results on Primes

1981 1) Assume RH and $F(\alpha) = o(\log T)$. Then

$$\psi(x) = x + o(x^{1/2} \log^2 x).$$

PNT error improved!

2) If $F(\alpha) \ll 1$ and RH,

$$p_{n+1} - p_n \ll p_n^{1/2} (\log p_n)^{1/2}$$

$p_{n+1} - p_n \ll x^{1/2}/\log x$

$$\sum_{p_n \leq x} (p_{n+1} - p_n)^2 \ll x \log^2 x$$

$$\sum_{\substack{p_n \le x \\ p_{n+1}-p_n \ge H}} (p_{n+1}-p_n) \ll \frac{x\log x}{H}$$

Thus as soon as $\frac{H}{\log x} \to \infty$ this is non-trivial

(RH alone is one $\log x$ larger)
selberg

—

If $\overset{RH}{} F(\alpha) \sim 1$ near $\alpha = 2$

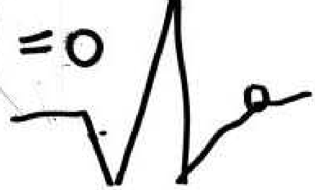
$$p_{n+1} - p_n = o\left(p_n^{1/2} (\log p_n)^{1/2}\right)$$

Heath-Brown Goldston

~

RH, $F(\alpha) \sim 1$ for α around 1

$$\liminf_{n\to\infty} \left(\frac{p_{n+1}-p_n}{\log p_n}\right) = 0$$



Interesting result of H-B:

<u>Thm</u> For $T \geq 2$

$$\left| \sum_{0<\gamma \leq T} x^{i\gamma} \right| \ll T^{1/2} \left(\max_{t \leq T} F(x,t) \right)^{1/2}$$

This is non-trivial if $F(x,T) = o(T \log^2 T)$

$$F(x,T) = \int_{-\infty}^{\infty} \Big| \sum_{0<\gamma\le T} x^{i\gamma} e^{i\gamma u} \Big|^2 e^{-2|u|} \, du$$

(related by Plancherel to other formula)

By Gallagher inequality

$$|f(0)| \ll \int_{-1}^{1} |f(u)| \, du + \int_{-1}^{1} |f'(u)| \, du$$

$f \in C^1$, Take

$$f(u) = \Big| \sum_{0<\gamma\le T} x^{i\gamma} e^{i\gamma u} \Big|^2$$

Then

$$\Big| \sum_{0<\gamma\le T} x^{i\gamma} \Big| \ll \int_{-1}^{1} \Big| \sum_{\gamma} x^{i\gamma} e^{i\gamma u} \Big|^2 du$$

$$+ \int_{-1}^{1} 2 \Big| \sum_{\gamma} x^{i\gamma} e^{i\gamma u} \Big| \Big| \frac{\partial}{\partial u} \sum_{\gamma} x^{i\gamma} e^{i\gamma u} \Big| du$$

First is

$$\ll \int_{-\infty}^{\infty} \Big| \sum_{\gamma} x^{i\gamma} e^{i\gamma u} \Big|^2 e^{-2|u|} du = F(x,T)$$