Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Gaussian ensembles of random matrices V

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The leading term of large-N asymptotics:

$$\int_\Gamma \varphi(z)\, e^{NF(z)}\, dz \simeq \varphi(z_0)\sqrt{\frac{2\pi}{N|F''(z_0)|}}\; e^{NF(z_0)+\frac{i}{2}\left(\pi - \operatorname{Arg}\frac{F''(z_0)}{2}\right)}$$

Now we apply this method to our integral

$$I_{N+K}(x) = \int_0^\infty dq\, q^K e^{Nf(q)}, \quad f(q) = \ln q - \frac{1}{2}(q - ix)^2$$

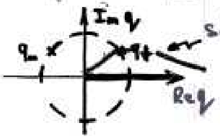
Saddle-point condition: $f'(q) = \frac{1}{q} - q + ix = 0$

Two solutions: $\begin{cases} q_+ = \frac{1}{2}\left(ix + \sqrt{4-x^2}\right) \\ q_- = \frac{1}{2}\left(ix - \sqrt{4-x^2}\right) \end{cases}$

Three different cases: i) $|x|<2$, ii) $|x|>2$, iii) $|x|=2$.

I) $|x|<2 \to x = 2\cos\varphi$, $0<\varphi<\pi$

Saddle points $q_\pm = i\cos\varphi \pm \sin\varphi$, or $\begin{cases} q_+ = e^{-i(\varphi - \frac{\pi}{2})} \\ q_- = e^{i(\varphi+\frac{\pi}{2})} \end{cases}$



Only q_+ is relevant, and

$\operatorname{Re} f(q_+) = \frac{1}{2}\cos(2\varphi)$, $\operatorname{Re} f(q)\big|_{q=0,\, q=\infty} \to -\infty$

hence both $q=0$ and $q=\infty$ are in negative sectors,

(in fact, in two <u>different</u> n.s.).

$$f''(q_+) = -(1+\frac{1}{q_+^2}) = 2i\sin\varphi\, e^{i\varphi} \to |C| = \sin\varphi,\ \theta = \varphi + \frac{\pi}{2}$$

Further using $I_{N+K}(-x) = \overline{I_{N+K}(x)}$ for real x, we get

$$h_{N+K}(x) \simeq N^{N+K}\sqrt{\frac{2}{\sin\varphi}}\, e^{\frac{N}{2}\cos 2\varphi} \cos\left\{(K+\tfrac{1}{2})\varphi - \frac{\pi}{4} + N\ \varphi - \tfrac{1}{2}\sin 2\varphi)\right\}$$

the required Plancherel-Rotach asymptotics

Bulk scaling regime for GUE: $|\lambda| < 2$, $N \to \infty$

1) Mean density of eigenvalues:

$$\overline{\rho_N(\lambda)} = e^{-\frac{N}{2}\lambda^2} \frac{1}{(N-2)!\,N^{N-1}} \left[h_N^2(\lambda) - h_{N-1}(\lambda) h_{N+1}(\lambda) \right]$$

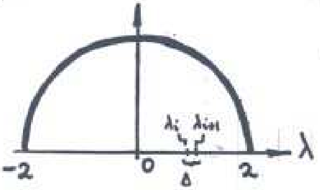
Parametrize $\lambda = 2\cos\varphi$ and exploit Plancherel-Rotach as.:

$$h_{N+K}(\lambda) \simeq N^{N+K} \sqrt{\frac{2}{\sin\varphi}}\, e^{\frac{N}{2}\cos 2\varphi} \cos(\alpha + K\varphi), \quad \alpha = \frac{\varphi}{2} - \frac{\pi}{4} + N\left(\varphi - \frac{\sin 2\varphi}{2}\right)$$

use $\cos^2\alpha - \cos(\alpha+\varphi)\cos(\alpha-\varphi) = \sin^2\varphi$, and $(N-1)! \simeq \sqrt{\frac{2\pi}{N}} N^N e^{-N}$

as well as $\exp\{-\frac{N}{2}\lambda^2\} = \exp\{-N(1+\cos 2\varphi)\}$, and find

$$\lim_{N\to\infty} \frac{1}{N} \overline{\rho_N(\lambda)} = \rho_\infty(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}, \quad |\lambda| < 2$$



Wigner semicircle

$$\Delta = \frac{1}{N\rho_\infty(\lambda)} = O\left(\frac{1}{N}\right)$$ mean level spacing

2) Kerne. $K(\lambda, \lambda'$ is similarly exp ssed as:

$$K_N(\lambda,\lambda') = e^{-\frac{N}{4}(\lambda^2+\lambda'^2)} \left[\tilde{h}_N(\lambda)\tilde{h}_{N-1}(\lambda') - \tilde{h}_N(\lambda')\tilde{h}_{N-1}(\lambda) \right] \frac{1}{\lambda - \lambda'}$$

Parametrize: $\lambda = 2\cos\varphi$, $\lambda' = 2\cos\varphi'$ and introduce $\Omega = (\varphi - \varphi')/2$, $\psi = (\varphi + \varphi')/2$

Then $\lambda - \lambda' = 4 \sin\Omega \sin\psi$.

Bulk scaling limit: $\lambda - \lambda' \propto \Delta = O\left(\frac{1}{N}\right) \to \boxed{\Omega = \frac{\omega}{N}}$, $\boxed{\omega = O(1)}$

$$\left[h_N(\lambda)h_{N-1}(\lambda') - h_N(\lambda')h_{N-1}(\lambda) \right] \propto \frac{e^{\frac{N}{2}[\cos 2\varphi + \cos 2\varphi']}}{\sqrt{\sin\varphi \sin\varphi'}} \left[\cos\alpha \cos(\alpha' - \varphi') - \cos\alpha' \cos(\alpha - \varphi) \right]$$

Number variance

consider the interval $B: (-\frac{L}{2}, \frac{L}{2})$, $-\frac{L}{2} \quad 0 \quad \frac{L}{2}$

$$\Sigma_2(L) = \overline{N^2_{(-\frac{L}{2},\frac{L}{2})}} - \overline{N}^2_{(-\frac{L}{2},\frac{L}{2})}.$$ We have shown

$$\Sigma_2(L) = \overline{N}_{(-\frac{L}{2},\frac{L}{2})} \quad \int \quad \int_{-L/2}^{L/2} d\lambda' \, Y_2(\lambda,\lambda')$$

Let the length L be of the order of $\Delta = \frac{1}{N\rho_\infty(0)}$,

so that $\rho(\lambda) \simeq \rho(\lambda') \simeq \rho_\infty(0)\cdot N$, $\lambda - \lambda' = r\Delta$, $L = s\Delta$

The number variance takes the form in the scaling limit

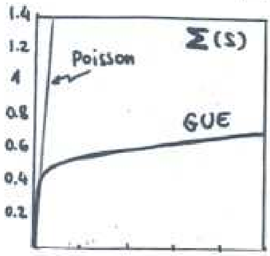
$$\Sigma_2(s) = s - 2\int_0^s dr\,(s-r)\left[\frac{\sin \pi r}{\pi r}\right]$$

$$\equiv \frac{2s}{\pi}\int_{\pi s}^{\infty} dx\, \frac{\sin^2 x}{x^2} \qquad 1-\cos x$$

In particular, large s behaviour

$$\Sigma_2(s \gg 1) = \frac{1}{\pi^2}\left[\ln 2\pi s + \gamma + 1\right] + O(1/s)$$

Euler's const.



Number variance for GUE spectrum vs. uncorrelated (poissonian) spectrum

In the large-N scaling limit

$$\begin{cases} \varphi = \psi + \omega/N \quad \varphi' = \psi - \frac{\omega}{N} \\ \beta = \psi - \frac{1}{2}\sin 2\psi \end{cases}$$

$\cos\alpha \cos(\alpha' - \varphi')$ $\quad \cos\alpha' \cos(\alpha\ \varphi)$

$$\simeq \cos\left[N\beta + \frac{\psi}{2} - \frac{\pi}{4} + 2\omega\sin^2\psi\right]\cos\left[N\beta - \frac{\psi}{2} - \frac{\pi}{4} - 2\omega\sin^2\psi\right]$$

$$\cos\left[N\beta + \frac{\psi}{2} - \frac{\pi}{4} - 2\omega\sin^2\psi\right]\cos\left[N\beta - \frac{\psi}{2} - \frac{\pi}{4} + 2\omega\sin^2\psi\right]$$

$$\sin\psi \ \ \underline{\sin(4\omega\sin^2\psi)}$$

with the same precision $\lambda\ \lambda \quad \frac{4\omega}{N}\pi\rho_\infty(\lambda)$,

$\cos 2\varphi_1 + \cos 2\varphi_2 \quad 2 \ \frac{\lambda^2}{2}\ 1)$ so that

$$h_N\ \lambda\ )h_{N-1}\ \lambda')\ \ h_N(\lambda')h_{N-1}(\lambda) \simeq 2N^{2N} e^{2N(\frac{\lambda^2}{2}-1)} \sin[\pi\rho_\infty(\lambda)N(\lambda-\lambda')]$$

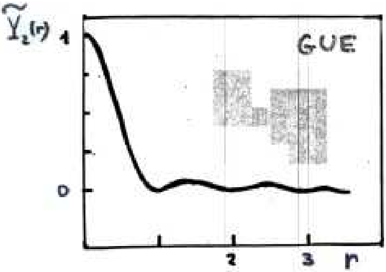
Finally we arrive to the asymptotic formula for the kernel

$$\lim_{N\to\infty} \frac{K_N\ \lambda,\lambda')}{K_N(\lambda,\lambda)} - K_\infty[N\rho_\infty(\lambda)(\lambda-\lambda')]$$

where $\boxed{K_\infty(r) = \frac{\sin\pi r}{\pi r}}$ Dyson kernel

Two point cluster function in the scaling limit

$$Y_2(\lambda,\lambda') \quad N^2\rho_\infty(\lambda)\rho_\infty\ \lambda\ )\ K_\infty^2[N\rho_\infty(\lambda)(\lambda-\lambda')]$$



Pseudocrystalline structure of random matrix spectra

$$\int_\infty dr\, \widetilde{Y}_2(r) \quad 1$$

II) "Spectral edge" regime for Hermite polynomials

$\underline{x=2}$ → Two saddle points q_+ q degenerate into one $\lim\limits_{x\to 2} q_+(x)$ $\lim\limits_{x\to 2} q_-(x)$ i → standard s.p. fails

Criterion of failure

Distance $|q_+ \; q_-$ ∼ "widths" $W \sim [N|f''(q_\pm)|]^{1/2}$

$\sqrt{4-x^2}$ ∼ $[N\sqrt{4-x^2}]^{-1/2}$

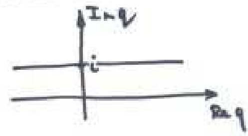
New regime starts at $\boxed{|x-2| \sim N^{-2/3}}$, when

$q_+ \; q_- \sim \sqrt{|2-x|} \sim N^{1/3} \longrightarrow \boxed{q = i + \frac{t}{N^{1/3}}}$ edge scaling

$\boxed{\xi = N^{\frac{2}{3}}(2-x)}$

Hermite polynomials

$$h_{N+K}(x) \propto \int_{-\infty}^{\infty} dq \; q^{N+K} e^{\frac{N}{2}(q- x)^2}$$



$$\frac{1}{N^{1/3}} \int dt \left(i+\frac{t}{N^{1/3}}\right)^K \exp N \ln\left(i+\frac{t}{N^{1/3}}\right)$$

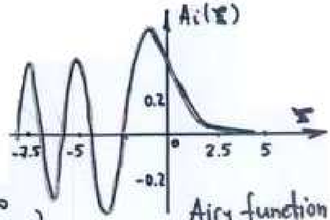
$$\frac{1}{2}\left[i+\frac{t}{N^{1/3}} \quad 2-\frac{\xi}{N^{2/3}}\right)\Big]\Big\}$$

Considering t, ξ finite when $N \to$ and expanding accordingly

$$h_{N+K}\left(x = 2 \;\; \frac{\xi}{N^{2/3}}\right) \quad \frac{N^{1/6}}{\sqrt{2\pi}} N^{N+K} e^{N \; N^{\frac{1}{3}}\xi} Ai(-\xi)$$

where

$$Ai \;\; \xi) \quad \frac{1}{\pi}\int^{\infty} dt \cos\left(\xi t + \frac{t^3}{3}\right)$$



Airy function

$$Ai(-\xi) \quad \begin{cases} \xi^{\frac{1}{4}} \frac{1}{\sqrt{\pi}} \cos\left(-\frac{2}{3}\xi^{3/2} + \pi/4\right), & \xi \to \infty \\ \frac{1}{\sqrt{\pi}} \frac{1}{|\xi|^{1/4}} e^{\frac{2}{3}|\xi|^{3/2}}, & \xi \to -\infty \end{cases}$$

$$Ai''(\xi) - \xi Ai(\xi) = 0$$

Edge scaling regime for GUE: $\lambda = 2 + \frac{\xi}{N^{2/3}}$, $N \to \infty$

density of eigenvalues:

$\rho \ \lambda \quad e^{-N\frac{\lambda^2}{2}} \tilde{D}_N(\lambda)$, $\tilde{D}_N(\lambda) = \langle \quad h_{N-1}(\lambda) h_{N+1}(\lambda)$

integral representation:

$$\int_{\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 \cdot \frac{q_1 - q_2}{q_1} e^{N[f(q_1) + f(q_2)]}$$

where $f(q) = \ln q - \frac{1}{2}(q - i\lambda$ as before.

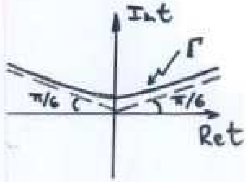
change to $q_{1,2}$, consider ξ, t_1, t_2 fixed

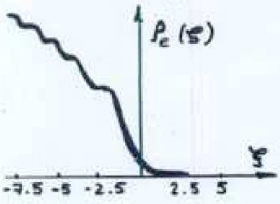
when $N \to \infty$, and find, to leading order

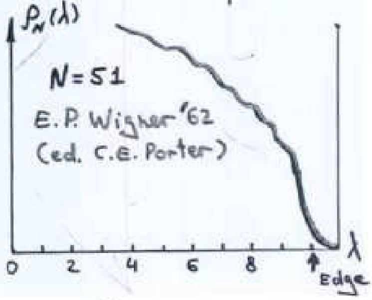
$$\overline{\rho_N\left(2 + \frac{\xi}{N^{2/3}}\right)} \propto \left\{ \left[Ai'(+\xi) \right]^2 - Ai''(+\xi) Ai(+\xi) \right\} \equiv \rho_e(\xi)$$

where: $Ai(\xi) = \frac{1}{\pi} \int_{\Gamma} dt \, e^{i(\xi t + \frac{t^3}{3})}$,

$$Ai'(\xi) = \frac{i}{\pi} \int_{\Gamma} dt \cdot t \, e^{i(\xi t + \frac{t^3}{3})}.$$







Kernel:

$$K(\xi_1, \xi_2) \propto \frac{Ai'(\xi_1) Ai(\xi_2) - Ai(\xi_1) Ai'(\xi_2)}{\xi_1 - \xi_2}$$, Tracy, Widom