Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Heuristic derivation of the n-point correlation function for the Riemann zeros II

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Lecture 11

3 and 4 point correlation functions

Main steps

- "Trace formula" for Riemann zeros: $s_j = \frac{1}{2} + i E_j$

$$d(E) = \bar{d}(E) + d^{(osc)}(E)$$

$$\bar{d}(E) = \frac{1}{2\pi} \ln \frac{E}{2\pi}$$

$$d^{(osc)}(E) = -\frac{1}{2\pi} \sum_n \frac{\Lambda(n)}{n^{1/2}} \left(e^{iE\ln n} + e^{-iE\ln n} \right)$$

- Correlation functions:

$$R_n^{(E)}(x_1, \dots, x_n) = \langle d(E+x_1)\, d(E+x_2) \dots d(E+x_n) \rangle_E$$

- GUE prediction $R_n^{(E)} \xrightarrow{E\to\infty} R_n^{(GUE)}$

$$R_n^{(GUE)}(x_1, \dots, x_n) = \det \left(\frac{\sin \pi \bar{d}(x_i - x_j)}{\pi (x_i - x_j)} \right)_{i,j=1\dots}$$

- Hardy-Littlewood conjecture (1923)

$$[\text{Number of prime pairs } p, q \text{ such that } p = q + h \text{ and } p \leq N] \xrightarrow[N\to\infty]{} \frac{N}{\ln^2 N} \alpha(h)$$

$$\alpha(h) = \sum_{\substack{(p,q)=1 \\ 1 \le p < q}} \left(\frac{\mu(q)}{\varphi(q)}\right)^2 e^{2\pi i \frac{p}{q} h} =$$

$$= 2\, C_2 \prod_{\substack{p \mid h \\ p>2}} \frac{p-1}{p-2} \qquad \text{if } h \text{ is even}$$

$$= 0 \qquad \text{if } h \text{ is odd}$$

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.66016..$$

- Two-point correlation function

$$\epsilon = x_1 - x_2$$

$$R_2(\epsilon) = \bar{d}^2 + R_2^{(diag)}(\epsilon) + R_2^{(off)}(\epsilon)$$

$$R_2^{(diag)}(\epsilon) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \epsilon^2} \ln\left[\, |\zeta(1+i\epsilon)|^2\, \Phi^{(diag)}(\epsilon) \right]$$

$$R_2^{(off)}(\epsilon) = \frac{1}{4\pi^2}\, \zeta(1+i\epsilon)^2\, e^{2\pi i \bar{d} \epsilon}\, \Phi^{(off)}(\epsilon)$$

When $\epsilon \to 0$

$$R_2(\epsilon) \to \bar{d}^2 - \frac{\sin^2 \pi \bar{d} \epsilon}{\pi^2 \epsilon^2} = \text{GUE result.}$$

Asymptotic formulas

$$\sum_{n\geq 1}^{\infty} \frac{1}{n}\left(\sum_h \alpha(h)\, e^{iE\frac{h}{n}}\right) e^{ix\ln n} \xrightarrow[\substack{E\to\infty \\ x\ln E = \text{const.}}]{}$$

$$\to \frac{e^{i\ln\frac{E}{2\pi}\cdot x}}{x^2}$$

$$\sum_{h=-\infty}^{+\infty} \alpha(h)\, e^{iE\frac{h}{y}} \longrightarrow (Z-\ln y)\,\theta(\ln y - Z)$$

$$Z = \ln\frac{E}{2\pi} \equiv 2\pi \bar{d}(E)$$

$$\theta(x) = \text{the step function} = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

Generalized Hardy-Littlewood conjecture (1923) (Conj. D.

$$\left[\text{Number of prime pairs } p, q \text{ such that } mp = nq + h \text{ and } p \leq N\right] \xrightarrow[N\to\infty]{} \frac{N}{\ln^2 N}\,\bar{\alpha}(h)$$

$$\bar{\alpha}(h) = \frac{1}{n}\, 2\cdot C_2 \prod_{\substack{p|h \\ p|m \\ p|n}} \frac{p-1}{p-2}\ ;$$

n coprime to n

$\boxed{h \text{ is coprime to } m, n, p, q}$

3 point correlation function

$$R_3^{(GUE} = \begin{vmatrix} \bar{d} & S(x_1,x_2) & S(x_1,x_3) \\ S(x_2,x_1) & \bar{d} & S(x_2,x_3) \\ S(x_3,x_1) & S(x_3,x_2) & \bar{d} \end{vmatrix}$$

$$= \bar{d}^3 - \bar{d}\, S(x_1 x_2) S(x_2 x_1) -$$
$$- \bar{d}\, S(x_1 x_3) S(x_3, x_1) - \bar{d}\, S(x_2 x_3) S(x_3, x$$
$$+ R_3^{(connected)},$$

$$R_3^{(connected)}(x_1, x_2, x_3) = \begin{vmatrix} 0 & S(x_1,x_2) & S(x_1,x_3) \\ S(x_2,x_1 & 0 & S(x_2,x_3) \\ S(x_3,x_1) & S(x_3,x_2) & 0 \end{vmatrix}$$

$$S(x_i, x_j) = \frac{\sin \pi \bar{d}(x_i - x_j)}{\pi (x_i - x_j)}$$

$$R_3^{(connected)}(x_1,x_2,x_3) = -\frac{1}{2\pi^3 (x_1-x_2)(x_2-x_3)(x_3-x_1)} \times$$
$$\times \left(\sin 2\pi\bar{d}(x_1-x_2) + \sin 2\pi\bar{d}(x_2-x_3) + \sin 2\pi\bar{d}(x_3-x_1) \right)$$

← GUE formula

From "Riemann" trace formula:

$$R_3^{(\text{Riemann})}(x_1,x_2,x_3) = -\frac{1}{4\pi^3}\,\mathrm{Re}\sum_{p_1}\sum_{p_2}\sum_{p_3}\frac{\ln p_1 \ln p_2 \ln p_3}{\sqrt{p_1 p_2 p_3}}\,\times$$

$$\Big[\left\langle e^{iE\ln(p_1p_2p_3)}\right\rangle_E e^{ix_1\ln p_1 + ix_2\ln p_2 + ix_3\ln p_3} +$$

$$+\left\langle e^{iE\ln\frac{p_1}{p_2p_3}}\right\rangle_E e^{ix_1\ln p_1 - ix_2\ln p_2 - ix_3\ln p_3} +$$

$$+\left\langle e^{iE\ln\frac{p_2}{p_1p_3}}\right\rangle_E e^{ix_2\ln p_2 - ix_1\ln p_1 - ix_3\ln p_3}$$

$$\left\langle e^{iE\ln\frac{p_3}{p_1p_2}}\right\rangle_E e^{ix_3\ln p_3 - ix_1\ln p_1 - ix_2\ln p_2}\Big]$$

3 contributions:

- $p_1 = p_2 p_3 + h$
- $p_2 = p_1 p_3 + h$
- $p_3 = p_1 p_2 + h$

From "Riemann" trace formula

$$R_3^{(\text{Riemann})}(x_1, x_2, x_3) = -\frac{1}{4\pi^3}\,\mathrm{Re}\sum_{p_1}\sum_{p_2}\sum_{p_3}\frac{\ln p_1 \ln p_2 \ln p_3}{\sqrt{p_1 p_2 p_3}}\,\times$$

$$\Big[\left\langle e^{iE\ln(p_1 p_2 p_3)}\right\rangle_E e^{i x_1 \ln p_1 + i x_2 \ln p_2 + i x_3 \ln p_3} +$$

$$+\left\langle e^{iE\ln\frac{p_1}{p_2 p_3}}\right\rangle_E e^{i x_1 \ln p_1 - i x_2 \ln p_2 - i x_3 \ln p_3} +$$

$$+\left\langle e^{iE\ln\frac{p_2}{p_1 p_3}}\right\rangle_E e^{i x_2 \ln p_2 - i x_1 \ln p_1 - i x_3 \ln p_3}$$

$$\left\langle e^{iE\ln\frac{p_3}{p_1 p_2}}\right\rangle_E e^{i x_3 \ln p_3 - i x_1 \ln p_1 - i x_2 \ln p_2}\Big]$$

3 contributions:

- $p_1 = p_2 p_3 + h$
- $p_2 = p_1 p_3 + h$
- $p_3 = p_1 p_2 + h$

$$e^{iE \ln \frac{p_1}{p_2 p_3}} \longrightarrow e^{i\frac{Eh}{p_2 p_3}}, \quad h \ll p_2 p_3$$

Probability that p_2 and p_3 are primes and $p_2 p_3 + h$ is also a prime

$$\left\langle \sum_{\substack{h \\ (h n_2) = (h n_3) = 1}} \Lambda(n_2)\Lambda(n_3)\Lambda(n_2 n_3 + h) e^{i\frac{Eh}{n_2 n_3}} \right\rangle_{n_2 n_3}$$

$$\approx (Z - \ln n_2 - \ln n_3)\,\Theta(\ln n_2 + \ln n_3 - Z)$$

$$-(Z - \ln n_3)\,\Theta(\ln n_3 - Z)$$

$$-(Z - \ln n_2)\,\Theta(\ln n_2 - Z)$$

$$Z = 2\pi \bar{d} \equiv \ln \frac{E}{2\pi}.$$

inclusion - exclusion principle

$$\sum_{\substack{(h, p_1) = 1 \\ (h, p_2) = 1}} f(h) = \sum_h f(h) - \sum_h f(h p_1) - \sum_h f(h p_2) + \sum_h f(h p_1 p_2)$$

$$3 \quad x_1 \quad x_2 \; x_3)\alpha \qquad 4\pi \quad Re \int^{\infty} d\alpha_2 \int^{\infty} d\alpha_3 \, e^{(x_1 \; x_2)\alpha_2 + (x_1 \; x_3)\alpha_3}$$

$$\left[\; Z \quad \alpha_2 \quad \alpha_3 \; \theta(\alpha_2 + \alpha_3 \; Z \quad (Z \; \alpha_3)\theta(\alpha_3 \; Z \quad (Z-\alpha_2)\theta(\alpha_2 - Z\right.$$

<u>Many different methods of computation</u>

E.g.

$$f(z) = \frac{\partial^2}{\partial z^2}\,(\text{Integrals}) \quad \int_0^{\infty} d\alpha_2 \int_0^{\infty} d\alpha_3 \left[\delta(Z \; \alpha_2 \; \alpha_3) \;\; \delta(Z-\alpha_2) - \delta(Z \; \alpha_3) \right] e^{x_{12}\alpha_2 \; x_{13}\alpha_3}$$

$$x_{12} \quad x_1 \; x_2, \quad x_{13} \quad x \quad x_3$$

$$g(z) = \int^{\infty} d\alpha_2 \, d\alpha_3 \, \delta(Z \; \alpha_2 \; \alpha_3) e^{x_{12}\alpha_2 \; x_{13}\alpha_3}$$

$$\int g(z) e^{zs} dz \qquad \int^{\infty} d\alpha_2 \int_0^{\infty} d\alpha_3 \, e^{\alpha_2 \, s \; x_{12}) - \alpha_3 (s \; x_{13})}$$

$$= \frac{1}{(s-x_{12})(s-x_{13})} \qquad \frac{}{x_{12} \quad x_{13}} \left(\frac{1}{s-x_{12}} - \frac{1}{s-x_{13}} \right)$$

$$\frac{1}{x_2 - x_3} \left(\frac{1}{s-x_{12}} \quad \frac{}{s \quad x_{13}} \right)$$

$$g(z \qquad \frac{}{x_2 \quad x_3} \left[e^{z x_{12}} \quad e^{z x_3} \right]$$

$$f(z \qquad \frac{}{x_{23}} \left[e^{z x_{12}} - e^{z x_{13}} \right] + \; e^{z x_{12}} \frac{}{x_{13}} + e^{z x_{13}} \frac{}{x_{12}}$$

$$= e^{z x_{12}} \left[\frac{}{x_1 \quad x_3} - \frac{}{x_2 \quad x_3} \right] + e^{z x_{13}} \left[\frac{}{x_1 \quad x_2} + \frac{1}{x_2 - x_3} \right]$$

$$e^{z x_{12}} \frac{(x_2 - x_1)}{(x_1 \quad x_3 \; (x_2 - x_3)} + e^{z x_{13}} \frac{(x_1 - x_3)}{(x_1 \quad x_2)(x_2 - x_3)}$$

$$d\alpha_2 \int_0^\infty d\alpha_3\, e^{(x_1 \; x_2)\alpha_2 \quad (x_1 \; x_3)\alpha_3}\, G(\alpha_2, \alpha_3)$$

$$\frac{e^{Z \; x \quad x_2}}{x_1 \; x_2 \; (x_2 \quad x_3)(x_3 - x_1)} + \frac{e^{Z(x_1 \quad x_3}}{(x_1 \quad x_3)(x_2 \quad x_3)(x_1 - x_2)}$$

$$r_3 \; x_1 \; x_2 \; x_3) \qquad \frac{1}{(2\pi)^3 i\,(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}$$

$$\left[e^{2\pi i \alpha (x_1 \quad x_2)} \qquad e^{2\pi i \alpha (x_1 - x_2} \quad - e^{2\pi\alpha (x_1 - x_3} \quad + e^{2\pi i (x_3 \quad x_1)} \right]$$

$$\frac{}{4\pi^3 (x_1 - x_2 \; (x_2 \quad x_3)(x_3 \quad x_1} \left[\sin 2\pi\alpha (x - x_2 + \sin 2\pi\alpha \right.$$

$$R_3^{\text{Riemann})} (x_1, x_2, x_3 = r_3 (x_1 \quad x_2 \quad x_3) + (1 \leftrightarrow 2) + (\leftrightarrow 3)$$

$$\frac{}{2\pi^2 \; x_1 \quad x_2 \; (x_2 \quad x_3 \; (x_3 - x_1)}$$

$$\left[\sin 2\pi\alpha \quad x_1 \quad x_2) + \sin 2\pi\alpha \quad x_3 \quad x_1) + \sin 2\pi\alpha (x_2 \quad x_3) \right.$$

exactly the GUE result for the 3 point correlation function

4 point correlation function

GUE

$$R_4^{\text{(connected}}\; x_1, x_2, x_3\; x_4 \quad \begin{vmatrix} 0 & S(x_1\; x_2 & S\; x_1\; x_3 & S(x_1\; x_4 \\ S(x_2\; x_1) & 0 & S(x_2\; x_3 & S(x_2\; x_4) \\ S(x_3\; x_1 & S(x_3\; x_2 & 0 & S(x_3\; x_4) \\ S(x_4, x_1 & S(x_4\; x_2 & S(x_4\; x_3) & 0 \end{vmatrix}$$

$$S(x\; , x_j \quad \frac{\sin \pi \bar{d}\,(x - x_j}{\pi\;\; x - x_j)}$$

$$R_4^{\text{Riemann}}\; x_1, x_2, x_3, x_4) \quad (2\pi)^4 \sum_{n_1 n_2, n_3, n_4} \frac{\Lambda\; n_1\; \Lambda(n_2) \Lambda\; n_3)\; \Lambda(n_4}{(n_1\, n_2\, n_3\, n_4\;\;{}^{1/2}}$$

$$\langle\; e^{i(E+\;\;)\ln n} + e^{i(E+x_1)\ln n_1})(e^{i(E+x_2)\ln n_2} + e^{\;\;(E+x_2)\ln n_2})$$

$$(e^{i(E+x_3)\ln n_1} + e^{i(E+x_3)\ln n_3})(e^{\,(E+x_4)\ln n_4} + e^{-\,(E+x_4}$$

Different types of terms

$$n \;\approx n_2\, n_3\, n_4 \;\; , \; n_2 \quad n_1 n_3 n_4 \quad n_3 \quad n_1 n_2 n_4 , n_4 \approx$$

$$n\; n_2 \approx n_3\, n_4 \qquad n_1 n_3 \sim n_2 n_4 , \; n_1 n \qquad n_2 n_3$$

4 point correlation function

GUE

(connected

$$R_4 \quad x_1, x_2, x_3 \; x_4 \qquad \begin{vmatrix} 0 & s(x_1\,x & s\; x_1\, x_3) & s(x_1\; x_4 \\ s(x_2\; x_1) & 0 & S(x_2\; x_3 & s(x_2\, x_4) \\ s(x_3\, x_1 & s(x_3\; x_2 & 0 & s(x_3\; x_4) \\ s(x_4, x_1 & s(x_4\; x_2 & s(x_4\; x_3) & 0 \end{vmatrix}$$

$$s(x\; , x_j \qquad \frac{\sin \pi \bar{d}\,(x - x_j}{\pi\; x - x_j)}$$

$$R^{\text{Riemann}} \quad x_1, x_2\; x_3, x_4 \qquad (2\pi)^4 \sum_{n_1 n_2 n_3 n_4} \frac{\Lambda\; n_1\; \Lambda(n_2)\,\Lambda\; n_3\;\; \Lambda(n_4}{(n_1\, n_2\, n_3\, n_4\;\;^{1/2}}$$

$$\langle\; e^{i(E+\;\;)\ln n} + e^{i(E+x_1)\ln n_1})(e^{i(E+x_2)\ln n_2} + e^{\;\;(E+x_2)\ln n_2})$$

$$(e^{i(E+x_3)\ln n_1} + e^{i(E+x_3)\ln n_3})(e^{\;(E+x_4)\ln n_4} + e^{-\;(E+x_4}$$

Different types of terms

) $\quad n. \approx n_2 n_3 n_4 \;, n_2 \quad n_1 n_3 n_4 \quad n_3 \quad n\, n_2 n_4, n_4 \approx$

$n\; n_2 \approx n_3 n_4 \qquad n_1 n_3 \sim n_2 n_4 \,, \; n_1 n \qquad n_2 n_3$

first type terms

$$n_1 = n_2 n_3 n_4 + h \qquad + \text{permutations.}$$

$$\sum_{(h,\, n_2 n_3 n_4)=1} \langle \Lambda(n_2)\Lambda(n_3)\Lambda(n_4)\, \Lambda(n_2 n_3 n_4 + h) \rangle e^{iE\frac{h}{n_2 n_3 n_4}} \equiv \underline{G(\alpha_2 \alpha_3 \alpha_4)} =$$

$$= (Z - \alpha_2 - \alpha_3 - \alpha_4)\,\theta(\alpha_2 + \alpha_3 + \alpha_4 - Z) - (Z - \alpha_3 - \alpha_4)\,\theta(\alpha_3 + \alpha_4 - Z)$$

$$- (Z - \alpha_3 - \alpha_2)\,\theta(\alpha_3 + \alpha_2 - Z) - (Z - \alpha_3 - \alpha_4)\,\theta(\alpha_3 + \alpha_4 - Z)$$

$$+ (Z - \alpha_2)\,\theta(\alpha_2 - Z) + (Z - \alpha_3)\,\theta(\alpha_3 - Z) + (Z - \alpha_4)\,\theta(\alpha_4 - Z)$$

$$\alpha_i = \ln n_i \qquad Z = 2\pi \bar{d} = \ln \frac{E}{2\pi}$$

$$\Gamma_4(x_1; x_2, x_3, x_4) = \frac{1}{(2\pi)^4} \int_0^\infty d\alpha_2 \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_4\, G(\alpha_2, \alpha_3, \alpha_4) \times$$

$$\times e^{i\alpha_2(x_1 - x_2) + i\alpha_3(x_1 - x_3) + i\alpha_4(x_1 - x_4)} = \text{tedious calculations}$$

$$= \frac{}{8\pi^4} \Bigg[\frac{\cos 2\pi \bar{d}\,(x_1 - x_2)}{(x_1 - x_3)(x_2 - x_3)(x_1 - x_4)(x_2 - x_4)} +$$

$$+ \frac{\cos 2\pi \bar{d}\,(x_1 - x_3)}{(x_1 - x_2)(x_3 - x_2)(x_1 - x_4)(x_3 - x_4)}$$

$$+ \frac{\cos 2\pi \bar{d}\,(x_1 - x_4)}{(x_1 - x_2)(x_4 - x_2)(x_1 - x_3)(x_4 - x_3)} \Bigg]$$

Total contributions of type I terms

$$R_4^{(I)}(x_1,x_2,x_3,x_4) = \Gamma_4(x_1 x_2 x_3 x_4) + (1\leftrightarrow 2) + (1\leftrightarrow 3) + (1\leftrightarrow 4)$$

$$\frac{}{4\pi^4}\sum^{4}\sum_{j>i} \frac{\cos 2\pi d(x_i - x_j)}{\prod_{\substack{k\neq \\ k\neq}} (x_i - x_k)(x_j - x_k)}$$

Second type terms $\boxed{P_1 P_2 \approx P_3 P_4}$ (+ permutations

- diagonal terms $P_1 = P_3$, $P_2 = P_4$; $P_1 = P_4$, $P_2 = P_3$

type I off-diagonal terms

$$P_1 P_2 = P_3 P_4 + h$$

Semi-diagonal terms

$$P_1 = P_3 \quad P_2 = P_4 + h \quad \text{or} \quad P_1 = P_4 \quad P_2 = P_3 + h$$

type II off-diagonal terms

$$P_1 = n P_2 + h_1$$
$$n P_2 = m P_4 + h_2$$

for arbitrary integer
m and n

<u>Diagonal terms</u> $\quad p_1 \; p_3 \quad p_2 \; p_4 \; + \quad = p_4 \quad p_2 = p_4$

$$\Gamma^{(diag}(x_1 \; x_2 \; x_3 \; x_4$$

$$(2\pi) \sum_n \Lambda^2 (n \; e^{(x_1 \; x_3)\ln n} + c.c. \sum_n \Lambda^2 (n \; (e^{(x_2 \; x_4)\ln n} + c.$$

$$+ (3 \leftrightarrow 4)$$

$$= \frac{1}{(2\pi)^4} \int^{\infty} \alpha \, d\alpha \left(e^{i\alpha(x_1 - x_3} + c.c. \right] \left[\int_0^{\infty} \alpha \, d\alpha \left(e^{\alpha \; x_2 - x_4} + \right] \right.$$

$$= \quad 4\pi^4 \left[\frac{}{(x_1 \; x_3)^2 \, (x_2 \; x_4{}^2} + \text{permutations} \right]$$

<u>Total diagonal</u>

$$R_4^{(diag)} \qquad 4\pi^4 \sum_{=2}^{4} \frac{1}{(x_1 - x_{\cdot})^2 \; x_J - x_J \,)^2}$$

$$J \neq$$

$$J \neq 1 \qquad J' > J$$

$$R_4^{(diag)} \quad 4\pi \left[\frac{}{(x_1 \; x_2)^2 (x_3 \; x_4{}^2} + \frac{}{(x \quad x_3)^2 (x_2 \; x_4)^2} + \frac{1}{(x_1 - x_4)^2 (x_2 - x_3)^2} \right]$$

Type I off-diagonal contributions

$p_1 \; p_2 \quad p_3 \, p_4$

$$\sum_h \langle \Lambda \, n_1 \; \Lambda \, n_2 \; \Lambda \, n_3) \Lambda \, n_4 \; e\left(\frac{h}{n_2 n_3}\right)_{n \, n_2 \; n_3 \, n_4} \rangle = G(\alpha \; \alpha_2 \; \alpha \; \alpha_4$$

coprime to $p_1 n_2 n_3 n_4$

$$\sum_{(h \; p_1 p_2 p_3 p_4)} f h = \sum_h f h) \sum_h f(p_1 h \; \sum_h f \; p_2 h \; \sum_h f \; p_3 \; \sum f p_4$$

$$+ \sum_h f(p p_2 h + \sum_h f(p_1 p_4 h +$$

$$\sum_h f(\; p_2 p_3 h)$$

$$\sum_h f \; p_1 p_2 p_3 p_4 h)$$

$\alpha \quad \log n \qquad \alpha + \alpha_2 = \alpha_3 + \alpha_4$

$$G \; \alpha \; \alpha_2 \; \alpha_3, \alpha_4 \quad (Z - \alpha_2 - \alpha_3) \theta \; \alpha \; \alpha_3 \; Z \quad (Z - \alpha_3) \theta (\alpha_3 \; Z) -$$

$$(Z \; \alpha \;) \theta (\alpha \; Z \; - (Z \; \alpha_2 \; \theta(\alpha_2 \; Z)$$

$$(Z - \alpha_4 \; \theta(\alpha_4 \; Z +$$

$$Z \; \alpha_3 + \alpha \; \theta \; \alpha_3 \; \alpha \; Z)$$

$$+ Z - \alpha_4 + \alpha \;) \theta (\alpha_4 \; \alpha_1 - Z \; + (Z - \alpha_3 + \alpha_2 \; \theta(\alpha_3 \; \alpha_2 \; Z)$$

$$+ Z \; \alpha_1 + \alpha_2) \theta (\alpha_4 - \alpha_3 \; Z$$

$$R_4^{(I\text{-off})}(x_1 x_2; x_3, x_4) = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int_0^\infty d\alpha_3 \int_0^\infty d\alpha_4 \; \delta(\alpha_1+\alpha_2-\alpha_3-\alpha_4)$$
$$G(\alpha_1,\alpha_2;\alpha_3,\alpha_4)\, e^{i\alpha_1 x_1 + i\alpha_2 x_2 - i\alpha_3 x_3} \;\times$$

Total contribution:

$$r_4^{(I\text{-off})}(x_1,x_2,x_3,x_4) = -\frac{1}{4\pi^4}\sum_{=1}^{4}\sum_{j>i} \frac{\cos 2\pi\bar{d}\,(x_i-x_j)}{\prod_{\substack{K\\ k\neq i,j}}(x_i-x_K)\prod_{j'}(x_j-x_{k})}$$

~~[illegible]~~

exactly as the first type terms: $(P_1 = P_2 P_3 P_4 + h)$

(not by accident)

Semi-diagonal contributions

$$r = P_3\,,\quad P_2 = P_4 + h$$

As for 2-point correlation function,

$$r_4^{(s.d.)} = \frac{}{8\pi^4}\,\mathrm{Re}\sum_{i=1}^{2}\sum_{j=3}^{4}\frac{e^{i\,2\pi\bar{d}\,(x_i-x_j)}}{(x_i-x_j)^2}\int_0^\infty \alpha\, e^{i(x_{i'}-x_{j'})\alpha}$$
$$= -\frac{1}{8\pi^4}\sum_{i=1}^{2}\sum_{j=3}^{4}\frac{\cos[(x_i-x_j)2\pi\bar{d}]}{(x_i-x_j)^2(x_{i'}-x_{j'})^2}$$

Total semi-diagonal:

$$R_4^{(\text{semi-diagonal})}(x_1,x_2,x_3,x_4) \simeq -\frac{1}{4\pi^4}\Bigg[\frac{\cos 2\pi\bar{d}\,(x_1-x_2)}{(x_1-x_2)^2(x_3-x_4)^2} + \frac{\cos 2\pi\bar{d}\,(x_1-x_3)}{(x_1-x_3)^2(x_2-x_4)^2} + \frac{\cos 2\pi\bar{d}\,(x_1-x_4)^2}{(x_1-x_4)^2(x_2-x_3)^2} + \frac{\cos 2\pi\bar{d}\,(x_2-x_3)}{(x_2-x_3)^2(x_1-x_4)^2}$$
$$+ \frac{\cos 2\pi\bar{d}\,(x_2-x_4)}{(x_2-x_4)^2(x_1-x_3)^2} + \frac{\cos 2\pi\bar{d}\,(x_3-x_4)}{(x_3-x_4)^2(x_1-x_2)^2}\Bigg];$$

Type II - off-diagonal contributions

most difficult) $\quad p_1 p_2 \approx p_3 p_4$

$$p = n\, p_3 + h \quad (h \;\; m\, n\, p \;\;) =$$

$$n\, p_2 = m\, p_3 + h_2 \quad (h_2 \;\; m\, n\, p_2 p_3) =$$

$$\exp\left(E \ln \frac{p_1 \;\; 2}{p_3 \;\; p_4} \right) \longrightarrow ex\left(E \frac{h_1}{n p_3} + E \frac{h_2}{m p_4} \right)$$

$$\sum_{(h_1 \;\; mn)} \Lambda \; p_1 \; \Lambda \; p_3 \; e^{E \frac{h}{n p_3}} \;\; \rangle = \frac{1}{n} (Z \;\; \alpha_3 \;\; \ln n)\, \theta(\alpha + \ln n \;\; Z)$$

$m p_1 \quad n p_3 + h \quad$ ignoring $(h \;\; mn)$

$$\sum_{(h_2 \;\; m} \langle \Lambda (p_2 \; \Lambda \; p_3) \rangle e^{E \frac{h_2}{m p_1}} = {}_m (Z \;\; \alpha_4 \; \ln m \;\; \theta(\alpha_4 + \ln m$$

$$r^{\mathrm{II}\,\mathrm{off}}_{(x_1 \;\; x_2 \;\; x_3 x} \quad \frac{1}{(2\pi)^4 \; m n} \int^{\infty} d\alpha \int d\alpha \int d\alpha_3 \int^{\infty} d\alpha_4$$

$$\delta(\alpha + \ln m - \alpha_3 \;\; \ln n \quad Z \;\; \alpha_3 \;\; \ln n \;\; \theta(\alpha \;\; Z + \ln n$$

$$Z \;\; \alpha_4 \;\; \ln m)\, \theta(\alpha + \ln m \;\; Z) \times \delta(\alpha_2 + \ln n \;\; \alpha_3 \;\; \ln m$$

$$e^{x \alpha + i x_2 \alpha_2 \quad x_3 \alpha_3 \quad x_4 \alpha_4}$$

$$\frac{1}{(2\pi)^4 \; m n} J_1 \;\; J_2$$

$$\int d\alpha \int_0^\infty d\alpha_3 \, \delta(\alpha_1 \ \alpha_3 + \ln m \ \ln n)(Z \ \alpha_3 \ \ln n)\,\theta(\alpha_3 + \ln n - Z)$$
$$\times e^{x_1 \alpha - i x_3 \alpha_3}$$

$$I_2 - \int^\infty d\alpha_2 \int_0^\infty d\alpha_4 \, \delta(\alpha_2 + \ln n \ \ \alpha_4 \ \ln m \ \ (Z \ \ \alpha_4 - \ln m$$
$$\theta(\alpha_4 + \ln m - Z \ \ e^{x_2 \alpha_2 - \ x_4 \alpha_4}$$

$$e^{i x_3 \ln n - i x_1 \ln m} \ \frac{e^{(x_1 \ \ x_3) Z}}{(x_1 \ \ x_3)^2}$$

$$J_2 = e^{i x_4 \ln m - \ x_2 \ln n} \ \frac{e^{Z(x_2 \ \ x_4)}}{(x_2 \ \ x_4)^2}$$

Total

$$6\pi \ \sum \frac{e^{Z(x_1 + x_2 \ \ x_3 - x_4)}}{mn \, (x_1 - x_3)^2 (x_2 - x_4)^2} \, e^{(x_4 - x_1)\ln n}$$

inclusion exclusion principle with respect to mn

$$\sum_{(h_1, mn)} f(h_1) = \sum_{\delta \, mn} f \ \delta h_1 \ \mu(\delta$$

$$\sum_{h_1, mn} \langle \Lambda(P_1 \ \Lambda(P_2)| \, e_{mP_1}^{} \ E^h_{nP_3 \ \ nP_3 + h} - \sum_{\delta | mn} {}_h \ {}^{Z - \alpha_3 \ \ln n + \ln \delta) \times} \ \theta(\alpha_3 \ \ln n \ \ Z \ \ \ln\delta) \, \mu(\delta)$$

$\rightarrow$ The same as $Z \rightarrow Z + \ln \delta$

$$J_1 \to e^{\ln n - i x_1 \ln m} \sum_{\delta | mn} \frac{e^{i(x_1-x_3)(Z+\ln\delta)}}{(x_1-x_3)^2} \mu(\delta)$$

$P_m \to$ primes divided m

$P_n \to$ " — " n

$$J_1 = \frac{e^{i x_3 \ln n - i x_1 \ln m}}{(x_1-x_2)^2} e^{iZ(x_1-x_3)} \prod_{P_m}\left(1-e^{+i(x_1-x_3)\ln P_m}\right) \times \prod_{P_n}\left(1-e^{+i(x_1-x_3)\ln P_n}\right)$$

$$J_2 = \frac{e^{\ln m - i x_2 \ln n}}{(x_2-x_4)^2} e^{iZ(x_2-x_4)} \prod_{P_m}\left(1-e^{+(x_2-x_4)\ln P_m}\right) \times \prod_{P_n}\left(1-e^{+i(x_2-x_4)\ln P_n}\right)$$

Total

$$\frac{1}{16\pi^4} e^{iZ(x_1+x_2-x_3-x_4)} \frac{1}{(x_1-x_3)^2(x_2-x_4)^2} \sum_{mn} \frac{1}{mn} \prod_{m}\left(1-e^{i(x_1-x_3)\ln P_m}\right) \times \left(1-e^{i(x_2-x_4)\ln P_m}\right)$$

$$\times \prod_{P_n}\left(1-e^{i(x_1-x_3)\ln P_n}\right)\left(1-e^{i(x_2-x_4)\ln P_m}\right) \times$$

$$\times e^{i(x_4-x_1)\ln m + i(x_3-x_2)\ln n}$$

Define

$$\rho\ \alpha, \beta_1, \beta_2) = \sum_{m=1}^{\infty} \frac{}{m^{1+i\alpha}} \prod_{p|m} (1-e^{i\beta_1 \ln p})(1-e^{i\beta_2})$$

$$m = p_1^{k_1} p_2^{k_2} \quad p_e^{k_e}$$

$$\sum_{k=1}^{\infty} \frac{1}{p^{k(1+i\alpha)}} = \frac{\overline{p^{1+i\alpha}}}{1-\frac{1}{p^{1+i\alpha}}} = \frac{1}{p^{1+i\alpha}-1}$$

$$\rho(\alpha, \beta_1, \beta_2) = \prod_p \left(1 + \frac{(1-e^{i\beta_1 \ln p})(1-e^{i\beta_2 \ln p})}{p^{1+i\alpha}-1}\right)$$

$$\ln \rho(\alpha, \beta_1, \beta_2) \simeq \sum_p \frac{1}{p^{1+i\alpha}} (1-e^{i\beta_1 \ln p})(1-e^{i\beta_2 \ln p})$$

$$\approx \int_1^{\infty} \frac{dp}{p \ln p} e^{-i\alpha \ln p} (1-e^{i\beta_1 \ln p})(1-e^{i\beta_2 \ln p})$$

$$= \int_0^{\infty} \frac{dt}{t} e^{-i\alpha t} (1-e^{i\beta_1 t})(1-e^{i\beta_2 t}) \ ; \ t = \ln p$$

$$\frac{\partial}{\partial\alpha} \ln \rho(\alpha, \beta_1, \beta_2) = -i \int_0^{\infty} dt\, e^{-i\alpha t} (1-e^{i\beta_1 t})(1-e^{i\beta_2 t})$$

$$= -\frac{1}{\alpha} + \frac{1}{\alpha-\beta_1} + \frac{1}{\alpha-\beta_2} - \frac{1}{\alpha-\beta_1-\beta_2}$$

$$\ln \rho(\alpha, \beta_1, \beta_2) = \ln \frac{(\alpha-\beta_1)(\alpha-\beta_2)}{\alpha(\alpha-\beta_1-\beta_2)}$$

$$\alpha, \beta\ \beta_2 \qquad \frac{(\alpha-\beta_1)(\alpha-\beta_2)}{\alpha(\alpha-\beta_1-\beta_2)}$$

In the first sum

$$\alpha \quad x_1 - x_4 \qquad \beta_1 \quad x \quad x_3 \qquad \beta_2 = x_2 \quad x_4$$

$$\rho - \frac{(x_3 \quad x_4)\ x_1 \quad x_2)}{(x_1-x_4)(x_3-x_2)}$$

In the second sum

$$\alpha \quad x_2 \quad x_3\ , \beta \quad x \quad x_3 \qquad \beta_2 = x_2 \quad x_4$$

$$\rho \quad \frac{(x_2-x\ )\ \ x_4-x_3)}{(x_2 \quad x_3)\ (x_4 \quad x_1)}$$

Total contribution of the type II off-diagonal terms

$$r^{\text{II off}}_{(x_1 x_2 x_3 x_4)} - \frac{(x_1-x_2)^2\,(x_3-x_4)^2}{8\pi^4\,(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2}$$

$$\times \cos 2\pi\bar{\rho}\,(\,x_1 \quad x_3 + x_2 \quad x_4)$$

+ permutations

$$R^{(\text{II-off})}(x_1, x_2, x_3 x_4) = \frac{1}{8\pi^4}\sum_{i=2}^{4} \frac{(x_1-x_i)^2(x_j-x_{j'})^2}{(x_1-x_i)^2(x_1-x_{j'})^2\,(x_i-x_j)^2(x_i-x_{j'})^2}$$

$$\times \cos 2\pi\bar{\rho}\,(\,x_1 + x_i - x_j - x_{j'})$$

$j > j' \neq 1\ ,\ j \neq i,\ j' \neq i$

Table of contributions to the 4-point correlation function

$n_1 = n_2 n_3 n_4 + h$	$\dfrac{\cos(2\pi \bar{d}(x_1-x_2))}{4\pi^4 (x_1-x_3)(x_1-x_4)(x_3-x_2)(x_4-x_2)}$ + perm
$n_1 = n_2, \ n_3 = n_4$	$\dfrac{1}{4\pi^4}\left[\dfrac{1}{(x_1-x_3)^2(x_2-x_4)^2} + \dfrac{1}{(x_1-x_4)^2(x_2-x_3)^2} + \dfrac{1}{(x_1-x_2)^2(x_3-x_4)}\right.$
$n_1 n_2 = n_3 n_4 + h$	$\dfrac{\cos(2n\bar{d}(x_1-x_2))}{4\pi^4 (x_1-x_3)(x_1-x_4)(x_3-x_2)(x_4-x_2)}$ + perm.
$n_1 = n_2, \ n_3 = n_4 + h$	$-\dfrac{\cos(2n\bar{d}(x_1-x_2))}{4\pi^4 (x_1-x_2)^2(x_3-x_4)^2}$ + perm.
$m n_1 = n n_3 + h_1$ $n n_2 = m n_4 + h_2$	$\dfrac{1}{8\pi^2}\dfrac{(x_1-x_2)^2(x_3-x_4)^2 \cos(2n\bar{d}(x_1-x_3+x_2-x_4))}{(x_1-x_3)^2(x_1-x_4)^2(x_3-x_4)^2(x_3-x_4)^2}$ + perm

+

Exactly coincides with the GUE 4-point correlation function