Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Pair correlation of zeros of the Riemann zeta-function and prime numbers IV

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Equivalence between Strong PCC and Primes

-m)

Thm Assume RH. If $0 < B_1 < B_2 < 1$ then

$$\int_1^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2 dx \sim \frac{1}{2} \delta X^2 \log 1/\delta$$

provided $X^{-B_2} \le \delta \le X^{-B_1}$

$$F(x,T) \sim \frac{T}{2\pi} \log T$$

holds in $X^{B_1}(\log X)^{-3} \le T \le X^{B_2}(\log X)^3$

and conversely

Thm (Assume RH) Then

a) $F(x,T) \sim \frac{T}{2\pi}\log T$, $T \le x \le T^A$

or $F(\alpha) \sim 1$ $1 \le \alpha \le m$ every fixed $A > 1$

b) $\int_0^X (\psi(x+h) - \psi(x) - h)^2\, dx \sim hX \log X/h$

for $1 \le h \le X^{1-\epsilon}$

are equivalent

This implies earlier results on primes except PNT error term

Sketch of proof. Let

$$G_\delta(t) = \left(\frac{\sin\frac{\delta}{2}t}{\frac{\delta}{2}t}\right)\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)\,a_n(x)}{n^{it}} - \frac{2x^{1-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)}\right)$$

$e^\delta = 1 + \frac{1}{T}$ so $\delta \sim \frac{1}{T}$

$$-2x^{1/2-it}\sum_{\gamma}\frac{x^{i\gamma}}{1+(t-\gamma)^2}$$

$$=\sum_{n=1}^{\infty}\frac{\Lambda(n)a_n(x)}{n^{it}}-\frac{2x^{1-it}}{(\frac{1}{2}+it)(\frac{3}{2}-it)}$$

$$+O\left(x^{-\frac{1}{2}}\log(|t|+2)\right)$$

$$+O\left(\frac{x^{-3/2}}{|t|+2}\right)$$

$$F(x,T)=\frac{2}{\pi}\int_0^T\left|\sum_{\gamma}\frac{x^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt$$

$$+O(\log^3 T)$$

$$\hat{G}_\delta(y) = \frac{2\pi}{\delta} \sum_{\left|y+\frac{\log n}{2\pi}\right| < \delta/2\pi} \Lambda(n) a_n(x)$$

$$- \frac{2\pi x}{\delta} \int_{xe^{-\delta/2}}^{xe^{\delta/2}} a_v(e^{-2\pi y}) \frac{dv}{v}$$

By Parseval $\int_{-\infty}^{\infty} |G_\delta(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{G}(y)|^2 dy$

(simplifying)

$$\int_0^\infty \left(\sum_{y<n\le y+\frac{y}{T}} \Lambda(n) a_n(x) - x \int_{xe^{-\delta}}^{x} a_v(y) \frac{dv}{v} \right)^2 \frac{dv}{v}$$

$$= \frac{4x\delta^2}{\pi} \int_0^\infty \left(\frac{\sin \frac{\delta}{2} t}{\delta/2\, t} \right)^2 \left| \sum_\gamma \frac{x^{i\gamma}}{1+(t-\gamma)^2} \right|^2 dt$$

$$+ O\left(\frac{\log^2 T}{T} \right)$$

so $\boxed{L(x,T) = R(x,T)}$

Abelian / Tauberian Thm
gives easily

$$F(x,T) \sim \frac{T}{2\pi} \log T$$

$$\iff R(x,T) \sim \frac{x}{T} \log T$$

For $L(x,T)$ we need to remove $a_n(x)$
But for $\frac{y}{T}$ not too large, in

$$\sum_{y \le n < y + \frac{y}{T}} \Lambda(n) a_n(x)$$

$a_n(x)$ is pretty close to $a_y(x)$
so replace it. This actually only
works for $T \le x \le T^{2-\epsilon}$, but enough to
get small gap between primes

$$\Rightarrow \int_0^x (\pi(y + \lambda \log x) - \pi(y))^2 \, dy = (\lambda + \lambda^2) x$$

Of course it is obvious that partial summation does this better. Thus

$$L(x,T) = 4x^3 \int_x^\infty H(y)\frac{dy}{y^5} + O\left(\frac{x}{T^{3/2}}\log^4 x\right)$$

where $H(x) = \int_0^x \left(\psi(y+\frac{y}{T}) - \psi(y) - \frac{y}{T}\right)^2 dy$

Difference above to get $H(x) \sim \frac{x^2 \log T}{2T}$

Separate argument (Saffari-Vaughan type) to get

$$\int_0^x (\psi(y+h) - \psi(y) - h)^2 dy$$

$$\int_x^{2x}$$

Selberg's Theory of $S(T)$ and $\log \zeta(\sigma+it)$

$$S(t) = \frac{1}{\pi} \arg \zeta(\tfrac{1}{2}+it)$$

(1946) unconditiona $\quad T^{\frac{1}{2}+\epsilon} \leq H < T$

$$\int_T^{T+H} |S(t)|^{2K}\, dt = c_K H (\log\log T)^K$$

$$+ O_K\left(H (\log\log T)^{K-\frac{1}{2}}\right)$$

where $\quad c_K \quad \dfrac{(2K)}{K}\left(\dfrac{1}{2\pi}\right)^{2K}$

2kth Gaussian moment

Actually (simplify $H=T$)

$$\int_0^T \left| S(t) + \frac{1}{\pi}\sum_{p<T^{\frac{1}{k}}} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt$$

$$\ll_k T$$

Idea of Proof

(1943) Normal Density of Primes

$$\int_0^T \left| \frac{\zeta'}{\zeta}(\sigma+it) \right|^2 dt , \quad \sigma = \frac{1}{2} + \frac{\beta}{\log T} + it$$

Recall Landau Handbuch explicit formula

$$\frac{\zeta'}{\zeta}(s) \qquad \sum_{n<x} \frac{\Lambda(n)}{n^s} + \frac{x^{1-s}}{-s} \quad \sum_{\rho} \frac{x^{\rho-s}}{\rho-s}$$

$$+ \sum_{n=}^{\infty} \frac{x^{-2n-s}}{2n+s}$$

$$S(t) \quad \frac{1}{\pi} \arg \zeta\left(\tfrac{1}{2}+it\right)$$

$$\frac{1}{\pi} \int_{1/2}^{\infty} \mathrm{Im} \frac{\zeta'}{\zeta}(\sigma+it)\, dt$$

$$\approx \frac{1}{\pi} \int_{1/2}^{\infty} \left(-\sum_{n \le x} \frac{\Lambda(n)}{n^\sigma} \mathrm{Im}(n^{-it}) \right) d\sigma$$

$$\frac{1}{\pi} \sum_{n \le x} \frac{\Lambda(n) \sin(t \log n)}{n^{1/2} \log n}$$

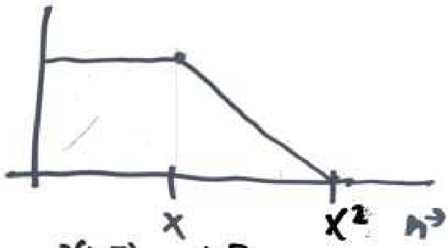
$$= -\frac{1}{\pi} \sum_{p \le x} \frac{\sin(t \log p)}{\sqrt{p}} + \text{prime powers}$$

Problem: $\sum_{\rho} \frac{x^{\rho-s}}{\rho-s}$ doesn't converge well enough

Selberg's 1^{st} explicit formula

Let $$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \le n \le x \\ \Lambda(n) \dfrac{\log \frac{x^2}{n}}{\log x} & \text{for } x \le n \le x^2 \end{cases}$$

Weight on $\Lambda(n)$ is



$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \le x^2} \frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)} - x^{1-s}}{\log x \,(1-x)^2}$$

$$+ \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2} + \text{trivial zeros term}$$

Sum over zeros, for $\sigma \geq \sigma_1$

Assume RH $\rightarrow$ $\ll \dfrac{x^{1/2-\sigma}}{(\sigma_1-1/2)\log x} \sum_{\gamma} \dfrac{(\sigma_1-1/2)}{(\sigma_1-1/2)^2+(t-\gamma)^2}$

This sum is by

partial fraction formula of $\frac{\zeta'}{\zeta}(s)$

$$= \mathrm{Re}\, \frac{\zeta'}{\zeta}(s)\Big|_{s=\sigma_1+it} + O(\log T)$$

Feed back explicit formula here and save

$\log x$ to get:

$$\sigma \geq \sigma_1$$

$$\frac{\zeta'}{\zeta}(\sigma+it) = -\sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma+it}}$$

$$+ O\left(x^{1/2-\sigma}\left|\sum_{n\leq x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\right)$$

$$O(x^{1/2-\sigma}\log t)$$

Integrate to get

Thm (RH) $t \geq 2$, $4 \leq x \leq t^2$, $\sigma_1 = \frac{1}{2} + \frac{1}{\log x}$,

$$S(t) = -\frac{1}{\pi} \sum_{n \leq x^2} \frac{\Lambda_x(n)}{n^{\sigma_1}} \frac{\sin(t \log n)}{\log n}$$

$$+ O\left(\frac{1}{\log x} \left| \sum_{n \leq x^2} \frac{\Lambda_x(n)}{n^{\sigma_1 + it}} \right| \right) + O\left(\frac{\log t}{\log x}\right)$$

$$= MT + \text{errors}$$

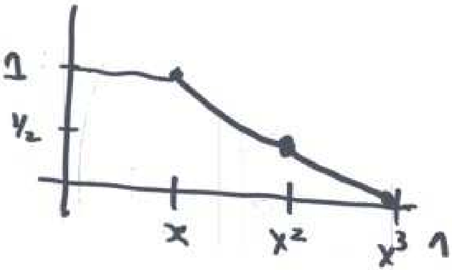
Lemma If $x = T^{\frac{1}{2k}}$

$$\int_0^T |MT|^{2k} dt = \frac{(2k)!}{k!\, 2^{2k}} T (\log\log T)^k$$

$$+ O\left(T (\log\log T)^{k-1}\right)$$

1946 Remove RH !

$$\Lambda_x(n) = \begin{cases} \Lambda(n), & 1 \le n \le x \\ \Lambda(n) \dfrac{\log^2 \frac{x^3}{n} - 2\log^2 \frac{x^2}{n}}{2\log^2 x}, & x \le n \le x^2 \\ \Lambda(n) \dfrac{\log^2 \frac{x^3}{n}}{2\log^2 x}, & x^2 \le n \le x^3 \end{cases}$$

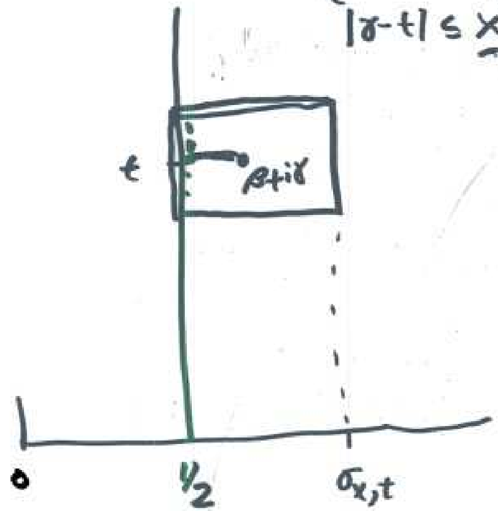


$$\frac{\zeta'}{\zeta}(s) = -\sum_{n \le x^3} \frac{\Lambda_x(n)}{n^s} + \cdots$$

$$\frac{1}{\log^2 x} \sum_{\rho} \frac{x^{\rho - s}(1 - x^{\rho - s})^2}{(s-\rho)^3} + \cdots$$

$$\sigma_{x,t} = \frac{1}{2} + 2\max_{\rho}\left(\beta - \tfrac{1}{2}, \frac{2}{\log x}\right)$$

$$|\gamma - t| \le \frac{x^{3|\beta - 1/2|}}{\log x}$$



Th

$$S(t) = -\frac{1}{\pi}\sum_{n \le x^3} \frac{\Lambda_x(n)}{n^{\sigma_{x,t}}}\,\frac{\sin(t\log n)}{\log n}$$

$$+ O\left((\sigma_{x,t} - \tfrac{1}{2})\left|\sum_{n^3 \le x} \frac{\Lambda_x(n)}{n^{\sigma_{x,t} + it}}\right|\right)$$

$$+ O\left((\sigma_{x,t} - \tfrac{1}{2})\log T\right)$$

After the 1990's, Selberg has published one paper (on this) in his collected works from the 1989 Amalfi Conference (which he had Bombieri deliver there)

This paper introduced the Selberg class of Dirichlet series which has attract a lot of interest — far more than for the theorems he proved in this class.

A. Ghosh 1981, 1983

$-1 < \lambda$ real

$$\int_0^T |S(t)|^\lambda \, dt \sim \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{\pi^{\lambda+1/2}} T (\log\log T)^{\lambda/2}$$

Example: $\lambda = 1$

Application to sign changes

Tsang Thesis 1984 describes Selberg's method for

$$\int_0^T F(\log \zeta(\sigma+it))\,dt$$

$F(z)=$ interesting functions like

$\operatorname{sgn}(\operatorname{Re} z)$

$\operatorname{sgn}(\operatorname{Im} z)$

$|\operatorname{Re} z|$

$|\operatorname{Im} z|$

Thm $\chi_{\alpha,\beta}(u)\ \begin{cases} & \alpha < u \le \beta \\ 0 & \end{cases}$

$$\int_0^T \chi_{\alpha,\beta}\left(\sqrt{\frac{\pi}{\log\log T}}\ S(t)\right)dt$$

$$= T\int_\alpha^\beta e^{-\pi u^2}\,du + O\left(\frac{T\log\log\log T}{\sqrt{\log\log T}}\right)$$

Also $\log \zeta(\sigma+it)$

Sign Changes of $S(t)$

1946 $\quad Z(\ )\quad \#\{\text{sign changes of } S(t) \text{ in } 0 < t \le T\}$

$$Z(T) \gg T(\log T)^{\frac{1}{3}} \exp(-c\log\log T)$$

Ghosh / unpublished Sel

$$Z(T) \gg T(\log T)^{1-\epsilon}$$

Th (1989) $\quad c > 0$

$$Z(T) \gg \frac{T\log T}{e^{c(\log\log\log T)^2}}$$

Used $\displaystyle\int_0^T \operatorname{sgn}(S(t))\operatorname{sgn}(S(t+h))\,dt$

$$T - \frac{2}{\pi}\frac{\sqrt{2c_1}\,T\log\log\log T}{\sqrt{\log\log T}}$$

$$h = \frac{R}{\log T}$$

$$R = e^{c(\log\log\log T)^3}$$

th

Thm

$$Z(T) \le \frac{c\,T\log T}{\sqrt{\log\log T}}\,(\log\log\log T)$$

If better error in $S(t)$ Gaussian

result get

Conjecture

$$Z(T) \sim \frac{T\log T}{\sqrt{\pi \log\log T}}$$