Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Gaussian ensembles of random matrices VI

Y. Fyodorov (Brunel)

2 April 2004

Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK

Tel: +44 1223 335999 Fax: +44 1223 330508
E-mail: webseminars@newton.cam.ac.uk
http://www.newton.cam.ac.uk/webseminars

Orthogonal polynomials

vs. characteristic polynomials

For a Hermitian matrix ensemble with p.d.f. $P(H) \propto e^{-\mathrm{Tr}\,Q(\hat{H})}$ consider statistics of the characteristic polynomials

$$d_N^{(H)}(\lambda) = \det(\lambda - \hat{H}) \equiv \prod_{i=1}^{N} (\lambda - \lambda_i)$$

In particular, consider the expectation value, $\left(d\mu(\lambda) \equiv e^{-Q(\lambda)} d\lambda \right)$

$$\overline{d_N^{(H)}(\lambda)} \propto \int d\mu(\lambda_1) \dots d\mu(\lambda_N) \cdot \prod_{i<j} (\lambda_i - \lambda_j)^2 \cdot \prod_{i=1}^{N} (\lambda - \lambda_i)$$

Observe:

$$\prod_{i<j}^{N} (x_i - x_j) \cdot \prod_{i=1}^{N} (x - x_i) \equiv \det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_N & x \\ x_1^2 & x_2^2 & \dots & x_N^2 & x^2 \\ \cdot & & \dots & & \\ x_1^N & x_2^N & \dots & x_N^N & x^N \end{pmatrix}$$

[Proof: the r.h.s. is a polynomial of degree N, with roots $x_1, x_2, \dots, x_N \longrightarrow$ equal to $C \cdot \prod_{i=1}^{N} (x - x_i)$, and must be $x^N \cdot (\text{van der Monde}) + \text{lower terms}$.]

We therefore can write:

$$\overline{d_N^{(H)}(\lambda)} \propto \int \prod_{i=1}^{N} d\mu(\lambda_i) \cdot \underbrace{\det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_N \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1} \end{pmatrix}}_{\prod_{i<j} (\lambda_i - \lambda_j)} \cdot \underbrace{\det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N & \lambda \\ & & \vdots & & \\ \lambda_1^N & \lambda_2^N & \dots & \lambda_N^N & \lambda^N \end{pmatrix}}_{\prod_{i<j} (\lambda_i - \lambda_j) \cdot \prod_{i=1}^{N} (\lambda - \lambda_i)}$$

$$= \sum_{\underset{\text{permutations of } (1,2,\dots,N)}{\mathcal{P}_s \in}} \mathrm{sign}(\mathcal{P}_s) \int \prod_{i=1}^{N} d\mu(\lambda_i) \cdot \lambda_{s_1}^{0} \lambda_{s_2}^{1} \dots \lambda_{s_N}^{N-1} \cdot \det \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N & \lambda \\ \lambda_1^N & \lambda_2^N & \dots & \lambda_N^N & \lambda^N \end{pmatrix}$$

$$= N! \int \prod_{i=1}^{N} d\mu(\lambda_i)\cdot \lambda_1^0 \lambda_2^1 \ \ \lambda_N^{N-1} \det \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N & \lambda \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^N & \lambda_2^N & \cdots & \lambda_N^N & \lambda^N \end{pmatrix}$$

$$= N! \int d\mu(\lambda_1)\ldots d\mu(\lambda_N) \det \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_N^{N-1} & 1 \\ \lambda_1 & \lambda_2^2 & & \lambda_N^N & \lambda \\ \lambda_1^2 & \lambda_2^3 & & \vdots & \vdots \\ \vdots & \vdots & & & \\ \lambda_1^{N-1} & \lambda_2^N & \cdots & \lambda_N^{2N-2} & \lambda^{N-1} \\ \lambda_1^N & \lambda_2^{N+1} & \cdots & \lambda_N^{2N-1} & \lambda^N \end{vmatrix}$$

$$\equiv N!\, D_N(\lambda)$$

where we introduced the polynomial of degree N

$$\det \left(\begin{array}{cccc|c} \int d\mu(\lambda) & \int d\mu(\lambda)\cdot\lambda & \cdots & \int d\mu(\lambda)\cdot\lambda^{N-1} & 1 \\ \int d\mu(\lambda)\cdot\lambda & \int d\mu(\lambda)\cdot\lambda^2 & \cdots & \int d\mu(\lambda)\cdot\lambda^N & \lambda \\ \vdots & & & \vdots & \vdots \\ \int d\mu(\lambda)\cdot\lambda^{N-1} & \cdots & \cdots & \int d\mu(\lambda)\cdot\lambda^{2N-2} & \lambda^{N-1} \\ \hline \int d\mu(\lambda)\cdot\lambda^N & \cdots & \cdots & \int d\mu(\lambda)\cdot\lambda^{2N-1} & \lambda^N \end{array} \right) \equiv D_N(\lambda)$$

Observe $\int d\mu(\lambda)\cdot\lambda^p D_N(\lambda) = 0$ for all $p = 0, 1, \ldots, N-1$

and also: $D_N(\lambda) = D_{N-1}\cdot\lambda^N + \ldots$, where $D_{N-1} = \det\left(\int d\mu(\lambda)\lambda^{i+j}\right)_{i,j=0}^{N} > 0$

So that $\int D_N^2(\lambda)\, d\mu(\lambda) = \int D_N(\lambda)\left[D_{N-1}\cdot\lambda^N + \overset{0}{\cancel{\ldots}}\right] = D_{N-1}\cdot D_N$

Conclusions:

I) $\dfrac{1}{\sqrt{D_{N-1} D_N}} D_N(\lambda) \equiv \Pi_N(\lambda)$ – explicit construction of orthonormal polynomials

II) $\overline{\det(\lambda - H)} \equiv \Pi_N^{m}(\lambda)$ – mean characteristic polynomial is orthogonal polynomial

monic

goes back to Heine-Borel, 1878

Let us now calculate:

$$\overline{\det(\lambda-H)\det(\lambda'-H)} \propto \int \prod_{i=1}^{N} d\mu(\lambda_i) \prod_{i<j}(\lambda_i-\lambda_j)^2 \prod_{i=1}^{N}(\lambda-\lambda_i)\prod_{i=1}^{N}(\lambda'-\lambda_i)$$

Notation: $\Delta_N(x_1,\dots,x_N) \equiv \det\begin{pmatrix} 1 & \cdots & 1 \\ x_1 & & x_N \\ x_1^{N-1} & & x_N^{N-1}\end{pmatrix}$ van der Monde

Observe: $$\Delta_{N+2}(x_1,\dots,x_N,x,x') = \prod_{i<j}(x_i-x_j)\prod_{i=1}^{N}(x-x_i)(x'-x_i)\cdot(x-x')$$

which allows us to write

$$\overline{d_N^{(H)}(\lambda)d_N^{(H)}(\lambda')} \propto \frac{1}{\lambda-\lambda'}\int\prod_{i=1}^{N}d\mu(\lambda_i)\cdot\Delta(\lambda_1,\dots,\lambda_N)\cdot\Delta(\lambda_1,\dots,\lambda_N,\lambda,\lambda')$$

$$\propto \frac{1}{\lambda-\lambda'}\int\prod_{i=1}^{N}d\mu(\lambda_i)\det\begin{pmatrix}\pi_0(\lambda_1) & \dots & \pi_0(\lambda_N)\\ \vdots & & \vdots \\ \pi_{N-1}(\lambda_1) & & \pi_{N-1}(\lambda_N)\end{pmatrix}\det\left(\begin{array}{ccc|cc}\pi_0(\lambda_1) & \dots & \pi_0(\lambda_N) & \pi_0(\lambda) & \pi_0(\lambda')\\ \vdots & & & & \vdots\\ \pi_{N+1}(\lambda_1) & \dots & \pi_{N+1}(\lambda_N) & \pi_{N+1}(\lambda) & \pi_{N+1}(\lambda')\end{array}\right)$$

$$\propto \frac{1}{\lambda-\lambda'}\int\prod_{i=1}^{N}d\mu(\lambda_i)\cdot\pi_0(\lambda_1)\pi_1(\lambda_2)\dots\pi_{N-1}(\lambda_N)\cdot\det\left(\begin{array}{cc|cc}\pi_0(\lambda_1) & \dots & \pi_0(\lambda) & \pi_0(\lambda')\\ \pi_1(\lambda_1) & \dots & \pi_1(\lambda) & \pi_1(\lambda')\\ \vdots & & \vdots & \vdots\\ \pi_{N+1}(\lambda_1) & \dots & \pi_{N+1}(\lambda) & \pi_{N+1}(\lambda')\end{array}\right)$$

$$\frac{1}{\lambda-\lambda'}\det\begin{pmatrix} \int d\mu\,\pi_0^2 & \cancel{\int d\mu\,\pi_1\pi_0}^{\,0} & \dots & \cancel{\int d\mu\,\pi_{N-1}\pi_0}^{\,0} & \pi_0(\lambda)\,\pi_0(\lambda') \\ \int d\mu\,\pi_0\pi_1 & \int d\mu\,\pi_1^2 & & \cancel{\int d\mu\,\pi_{N-1}\pi_1}^{\,0} & \pi_1(\lambda)\,\pi_1(\lambda') \\ \vdots & \vdots & & \int d\mu\,\pi_{N-1}^2 & \pi_N(\lambda)\,\pi_N(\lambda') \\ \cancel{\int d\mu\,\pi_0\pi_{N+1}}^{\,0} & \cancel{\int d\mu\,\pi_1\pi_{N+1}} & & \cancel{\int d\mu\,\pi_{N-1}\pi_{N+1}}^{\,0} & \pi_{N+1}(\lambda)\,\pi_{N+1}(\lambda') \end{pmatrix}$$

Finally,

$$\boxed{\overline{\det(\lambda-H)\det(\lambda'-H)} \propto \frac{\det\begin{pmatrix}\pi_N(\lambda) & \pi_N(\lambda')\\ \pi_{N+1}(\lambda) & \pi_{N+1}(\lambda')\end{pmatrix}}{\lambda-\lambda'}}$$

can find $\overline{\det(\lambda_1-H)\dots\det(\lambda_K-H)}$ in the same way.

Brezin, Hikami 2000; see also Mehta, Normand 2001
Forrester, Witte 2001

Important object $\overline{\left(\frac{\det(\lambda-H)}{\det(\nu-H)}\right)} \propto \int \prod_i d\mu(\lambda_i) \prod_{i<j}^{N} (\lambda_i-\lambda_j) \prod_{i=1}^{N} \frac{\lambda-\lambda_i}{\nu-\lambda_i}$

Notice $\prod_{i=1}^{N} \frac{1}{\nu-\lambda_i} = \sum_{i=1}^{N} \frac{1}{\nu-\lambda_i} \prod_{k\neq i} \frac{1}{\lambda_k-\lambda_i}$, and rewrite

$$\overline{\left[\frac{\det(\lambda-H)}{\det(\nu-H)}\right]} \propto N \int \prod_{i=1}^{N} d\mu(\lambda_i) \prod_{i<j} (\lambda_i-\lambda_j)^2 \cdot \prod_{i=1}^{N} (\lambda-\lambda_i) \cdot \frac{1}{\nu-\lambda_1} \frac{1}{\prod_{i=2}^{N}(\lambda_i-\lambda_1)}$$

further use: $\prod_{i=1}^{N}(\lambda_i-\lambda_j)^2 = \prod_{i=2}^{N}(\lambda_1-\lambda_i)^2 \prod(\lambda_i-\lambda_j)^2$, so that

$$\overline{\left(\det(\lambda-H)\right)} \propto \int \frac{d\mu(\lambda_1)}{\nu-\lambda_1} (\lambda-\lambda_1) \underbrace{\int \prod_{i=2}^{N} d\mu(\lambda_i) \prod_{2\le i<j}^{N} (\lambda_i-\lambda_j)^2 \prod_{i=2}^{N} (\lambda_1-\lambda_i)(\lambda-\lambda_i)}_{\overline{D_{N-1}^{(H)}(\lambda)\, D_{N-1}^{(H)}(\lambda_1)}}$$ ← Brezin, Hikami

$$\propto \int \frac{d\mu(\lambda_1)}{\nu-\lambda_1} \det\begin{pmatrix} \pi_{N-1}(\lambda) & \pi_{N-1}(\lambda_1) \\ \pi_N(\lambda) & \pi_N(\lambda_1) \end{pmatrix}$$

"Cauchy" transform

$$\boxed{\det\begin{pmatrix} \pi_{N-1}(\lambda) & f_{N-1}(\nu) \\ \pi_N(\lambda) & f_N(\nu) \end{pmatrix}}, \quad \Big\{ f_i \qquad \boxed{\int \frac{d\mu(\lambda)}{x-\lambda} \pi_N(\lambda)}$$

Similarly, one can find $\left[\overline{\frac{\qquad}{\det(\nu_1-H)\dots\det(\nu_2-H)}}\right]$

Fyodorov, Strahov 2003

see also

Baik, Deift, Strahov 2003

1) Relation to Riemann-Hilbert

2) New types of kernels, e.g.

$$W(\lambda,\lambda') = \frac{f_N(\lambda)\pi_{N-1}(\lambda') - f_{N-1}(\lambda)\pi_N(\lambda')}{\lambda-\lambda'}$$

3) applications