Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Heuristic derivation of the n-point correlation function for the Riemann zeros III

E.B. Bogomolny (Paris Sud)

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Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK

Tel: +44 1223 335999 Fax: +44 1223 330508
E-mail: webseminars@newton.cam.ac.uk
http://www.newton.cam.ac.uk/webseminars

ecture 3

n-point correlation function

Properties of permutations

$x_1 \; x_2 \quad x_n$ n different numbers

Define

$$\pi_j \, x \qquad \underset{\substack{k= \\ k\neq j}}{} \; \overline{x_j \;\; x_k}$$

Then

- $$\sum_{j=1} x_j \;\; \pi_j = - \begin{cases} 0 & 0 \le \quad \le n-2 \\ 1 & n \;\; 1 \end{cases}$$

- $$\sum_{j=1}^{n} \frac{1}{y \;\; x_j} \pi_j \, x - \prod_{k}^{n} \frac{1}{y \;\; x_k}$$

- $$\sum_{j=1} \quad x_j \;\; \pi_j(x) \quad \{ (\;)^{m-n} \quad m = n-1$$

- $$S_n \; y, z \equiv \sum_{\substack{\text{permutations} \\ \text{of } x_j}} (y \;\; x)(x_1 \;\; x_2)(x_2 \;\; x_3) \quad (x_{n-1} \, x_n)(x_n$$

$$S_n \; y, z - (y \;\; z^{\,n \;\; 1} \prod^{n} \frac{1}{(y \;\; x_k \;\; x_k \;\; z}$$

$$S_n(y, z = \; y \;\; z \sum_{[\gamma]\;[\delta]=n} \prod_{\{\gamma\}} \overline{y \;\; x_\gamma} \prod_{\{\delta\}} x_\delta^{\,1} \;\; z$$

all of n x_i into 2 $\{\gamma\}$ $\{\delta\}$

Proof

all x_j are inside the contour

$$J = \frac{1}{2\pi} \oint \frac{z}{\prod_{i=} (z \; x_i} dz \quad \sum_j x_j^m \prod_{k \neq j} \frac{1}{x_j \quad x_k}$$

hen $m \leq n-2$ integral is zero at ∞. When $n-$ has a pole at ∞

$$\to J \begin{cases} 0 & \leq 2 \\ 1 & 1 \end{cases}$$

$$\sum \frac{1}{y \quad x_j} \pi_j (x \qquad 2\pi i \oint \frac{dz}{(y-z) \prod_{k=1}^{n} (z - x_k)}$$

all x_j are inside but y outside

y

$$\prod_{k=} \overline{(y \quad x_k}$$

■

$S_n \; y \; z$ by induction..

$$S \; y \; z \quad \sum_j \frac{1}{(y \quad x_\cdot) \; x_j \quad z)} \; \vec{S}_{-2} \; x_i \quad x_j$$

$$\sum_{j}^{i \neq j} \frac{1}{(y - x_i)(x_j - z} \; (x \quad x_j) \; {}^s \prod_{p \neq i,j} \frac{1}{(x_i - x_p)(x_p - x_j)}$$

$$(-) \sum_{i,j} \overline{(y \quad x_i)} \prod_{\substack{d \\ d \neq}}^{n} \overline{x_i \quad x_d} \quad x_i \quad x_j \; {}^{n-1}_{z - x_j} \prod_{\substack{\beta \\ \beta \neq j}} \overline{x_j \quad x_\beta}$$

$$(-)^n \sum_{i,j} \overline{y \quad x_i} \; \pi_\cdot (x \quad x_i \quad x_j \quad \pi_j (x \; \frac{1}{z \quad x_j}$$

$$(x_i - x_j)^{n-} = (x_i - y + y - z + z - x_j)^{n-1} =$$

$$= \sum_{n_1+n_2+n_3=n-1} \frac{(n-1)!}{n_1!\, n_2!\, n_3!} (x_i - y)^{n_1} (y-z)^{n_2} (z - x_j)^{n_3}$$

only term with $n_1 = n_3 = 0$ $n_2 = n-1$ gives non-zero result

$$S_n(y,z) = (y-z)^{n-} \prod_{k=1}^{n} \frac{1}{(y-x_k)(x_k-z)}$$ ■

$$\frac{}{(y-x_k)(x_k-z)} = \left(\frac{1}{y-x_k} + \frac{1}{x_k-z}\right)\frac{1}{y-z}$$

$$S_n(y,z) = \frac{1}{y-z} \sum_{[\gamma]+[\delta]=n} \prod_{\{\gamma\}} \frac{1}{y-x_\gamma} \prod_{\delta} \frac{1}{x_\delta - z} \;;$$ ■

Example $n=2$. (x_1, x_2) (x_2, x_1)

$$\frac{1}{(y-x_1)(x_1-x_2)(x_2-z)} + \frac{1}{(y-x_2)(x_2-x_1)(x_1-z)} =$$

$$= \frac{1}{(x_1-x_2)}\left[\frac{1}{(y-x_1)(x_2-z)} - \frac{1}{(y-x_2)(x_1-z)}\right] = \frac{1}{(x_1-x_2)}\left[\frac{yx_1 - yz - x_2x_1 + x_2 z - yx_2 + yz + x_1x_2}{(y-x_1)(x_1-z)(y-x_2)(x_2-z)}\right]$$

$-x_1 z$

$$(y-z)\frac{1}{(y-x_1)(x_1-z)(y-x_2)(x_2-z)} = \frac{}{(y-z)}\left[\frac{1}{y-x_1} + \frac{1}{x_1-z}\right]\left[\frac{1}{y-x_2} + \frac{1}{x_2-z}\right] =$$

$$= \frac{1}{(y-z)}\left[\frac{}{(y-x_1)(y-x_2)} + \frac{1}{(y-x_1)(x_2-z)} + \frac{1}{(x_1-z)(y-x_2)} + \frac{1}{(x_1-z)(x_2-z)}\right]$$

(12), (0) (1) (2) (2) (1) (0) (12).

n-point correlation function for GUE

(GUE)

$$(x_1, x_2, \dots, x_n) = \det_{n\times n} \frac{\sin \pi \bar{d}\,(x_i - x_j)}{\pi (x_i - x_j)} \Big)$$

- **Determinant = sum over cycles**

$$R_n^{GUE} = \sum_{\text{cycles}} (-1)^{n-m} \prod_{j=1}^{m} \left[S(x_1^j, x_2^j)\, S(x_2^j x_3^j) \dots S_{n_j}(x^j_{n(j)}, x_1) \right]$$

$$\sum_{j=1}^{n} n(j) = n$$

$$\begin{pmatrix} 1 & 2 & & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & \dots & n_1 \\ \sigma_1 & \sigma_2 & & \sigma_{n_1} \end{pmatrix} \begin{pmatrix} n_1+1 & \dots & n_2 \\ \dots & & \end{pmatrix} \dots \begin{pmatrix} & \end{pmatrix}}_{m \text{ cycles}}$$

Example $n=3$

$$\underset{m=1}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}} \quad \underset{m=1}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}} \Big| \underset{m=2}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix}} \Big| \underset{m=2}{\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}}, \underset{m=2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}} \Big| \underset{m=3}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix}}$$

$$\det \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = (-1)^{3-1} S_{12} S_{23} S_{31} + (-1)^{3-1} S_{13} S_{32} S_{21} + (-1)^{3-2} S_{12} S_{21} S_{33}$$
$$+ (-1)^{3-2} S_{13} S_{31} S_{22} + (-1)^{3-2} S_{23} S_{32} S_{11} + (-1)^{3-3} S_{11} S_{22} S_{33}$$

- **Consider one cycle for N points**

$$C_N = S(x_1, x_2)\, S(x_2\, x_3) \quad S(x_N, x_1)$$

$$x_i, x_j) = \frac{\sin \pi \bar{d}\,(x_i - x_j)}{\pi (x_i - x_j)} \rightarrow$$

$$\rightarrow \frac{1}{2\pi i} \sum_{\sigma = \pm} e^{\frac{i\pi \bar{d}\,(x_i - x_j)\sigma}{\sigma}} \cdot \frac{1}{(x_i - x_j)}$$

$$C_N = \frac{1}{(2\pi i)^N} \frac{1}{(x_1-x_2)(x_2-x_3)\dots(x_N-x_1)} \times$$

$$\times \sum_{\sigma_i = \pm 1} \sigma_1 \quad \sigma_N \exp\Big(i\pi \bar{d} \big[x_1(\sigma_1-\sigma_N) + x_2(\sigma_2-\sigma_1) + \dots + x_N(\sigma_N - \sigma_{N-1}) \big] \Big)$$

$\underline{\text{String of } \sigma_j}$

$\sigma_m = +1 \quad \alpha_i \le m < \beta_i$

$\sigma_m = -1 \quad \beta_i \le m < \alpha_{i+1}$

2r changes of sign

$\alpha_1 < \beta_1 < \alpha_2 \dots < \alpha_r < \beta_r$

$$\underset{\alpha_1}{+} + \dots + \underset{\beta_1}{-} - - - \underset{\alpha_2}{+} + + \dots + \underset{\beta_2}{-} -$$

(= phase) depends only on $\alpha_i \ \beta_i$

$\mu_i = \beta_i - \alpha_i - 1$ (= number of +), $\nu_i = \alpha_{i+1} - \beta_i - 1$ (number of −)

$C_N(x_{\alpha_1} \quad x_{\alpha_2} \dots x_{\beta_2} \quad =$

$$= \frac{(-1)}{(2\pi i)^N} \prod_{i=1}^{r} \underbrace{\frac{1}{(x_{\alpha_i} - x_{\alpha_i+1})\dots(x_{\alpha_i+\mu_i} - x_{\beta_i})}}_{\text{positive}} \times$$

$$\times \prod_{i=1}^{r} \frac{1}{(x_{\beta_i} - x_{\beta_i+1})\dots(x_{\beta_i+\nu_i} - x_{\alpha_{i+1}})} \times$$

$$\times \exp\Big(2\pi i \bar{d}\big(\underbrace{x_{\alpha_1} - x_{\beta_1} + x_{\alpha_2} - x_{\beta_2} \dots + x_{\alpha_r} - x_{\beta_r}}_{\text{variables: phase variables}} \big)\Big)$$

all other − non-phase

Fix phase variables and consider all permutations of non-phase variables of one cycle

① permutation inside one positive block.

$$a_{\alpha_i \beta_i} \sum_{\text{permutations}} \frac{}{(x_\alpha \; x_{\alpha+1}(x_{\alpha+1} \; x_{\alpha+2} \quad x_{\alpha\,\mu} \; x_\beta)}$$

$\alpha \quad x_{\alpha} \quad x_{\alpha_i\,\mu}$

$$\frac{}{(x_\alpha \; x_\beta)} \sum_{\{\gamma\}+\{\delta\}} \prod_{\{\gamma\}}^{\mu} \frac{}{x_\alpha \; x_\gamma} \prod_{\delta} \frac{}{x_\delta \; x_\beta}$$

② Sum over permutations within each positive and negative blocks

$$\frac{(-\;)^{r}}{(2\pi i)^N} \frac{1}{(x_{\alpha_1} - x_{\beta_1})(x_{\beta_1} - x_{\alpha_2})\; x_{\alpha_2} - x_{\beta_2}) \quad (x_{\beta_r} \; x_{\alpha_r}}$$

$$\sum_{\substack{[\gamma_1] \; [\delta_1 \quad \mu \\ [\gamma_1'] + [\delta \;] \quad \nu_1 \\ \gamma_r] \quad \delta_r \quad \mu_r \\ \gamma_r] \; [\delta_r \quad \nu_r}} \prod \prod_{\gamma_i} \frac{}{(x_{\alpha_j} \; x_{\gamma_i})} \prod_{\delta_i} (x_\delta \quad x_{\beta_j}) \prod_{\gamma} \frac{}{(x_{\gamma_i'} - x_{\beta_i})}$$

$$\prod_{\delta'} \frac{1}{x_{\alpha_{i+1}} \quad x_{\delta_i}}$$

$$\exp\; 2\pi i\, d\,(x_{\alpha_1} \; x_{\beta_1} + x_{\alpha_2} \; x_{\beta_2} \quad + x_{\alpha_r} \; x_{\beta_r})$$

- Sum over all permutations of non-phase variables between different blocks

$$a_N^{(r)} = \frac{(-1)^r 2^{N-2r}}{(2\pi i)^N} \frac{1}{(x_{\alpha_1} - x_{\beta_1}) \quad (x_{\beta_r} - x_{\alpha_1})} \times$$

$$\times \sum_{[\gamma_1]+[\delta_1]+\dots+[\gamma_{2r}]+[\delta_r] = N-2r} \prod_{i=1}^{r} \prod_{\gamma_i} \frac{1}{(x_{\alpha_i} - x_{\gamma_i})} \prod_{\delta_i} \frac{1}{(x_{\delta_i} - x_{\beta_i})}$$

→ 2^{N-2r} – because each non-phase variable can be in $(+)$ or in (-1)

→ Summation is taken over all partition of $N-2r$ variables into $2r$ groups. Total → $(2r)^{N-2r}$ terms

- Terms with $r=0$ (all + or all −)

$$a_N^{(0)} = \frac{1}{(2\pi i)^N} \sum \frac{1}{(x_1-x_2)(x_2-x_3)\dots(x_N-x_1)}$$

Sum over all permutations of $1,2,\dots,N$

$$= \frac{1}{(2\pi i)^N} \sum_i S_{N-1}(x_i, x_i) = \frac{(-1)^{N-1}}{(2\pi i)^N} \sum_{i=1}^{N} (x_i - x_i) \prod_{\substack{j=1 \\ j \neq l}}^{N-2} \frac{1}{x_j - x_l} =$$

$$= \begin{cases} 0 & \text{if } N>2 \\ \dfrac{1}{2\pi^2 (x_1-x_2)^2} & \text{if } N=2. \end{cases}$$

Cycles with $r=0$ exist only for $N=2$.

- Multicycle contributions

Set of the phase variables $(x_{\alpha_i}, x_{\beta_i})$ can be split between different cycles

N ≠ 2

Cycle j contributes r_j variables to the exponent

non-phase variables

$$\underbrace{\sum_{[\gamma_1]+[\delta_1]+\dots+[\gamma_r]+[\delta_r]=n-2r}}_{\substack{\text{all partitions of } n-2r \text{ variables (non-phase)}\\ \text{into } 2r \text{ groups}}} \prod_{i=1}^{r} \prod_{\gamma_i} \frac{1}{(x_{\alpha_i}-x_{\gamma_i})} \prod_{\delta_i} \frac{1}{(x_{\delta_i}-x_{\beta_i})}$$

Remains the sum over cycles of phase variables

$$F(x_{\alpha_1}, x_{\beta_1} \dots x_{\alpha_r}\, x_{\beta_r}) = \sum_{\text{cycles}} (-1)^{2r-m} \times$$

$$\prod_{j=1}^{m} \frac{1}{(x^{(j)}_{\alpha_1}-x^{(j)}_{\beta_1})(x^{(j)}_{\beta_1}-x^{(j)}_{\alpha_2})\dots(x^{(j)}_{\beta_r}-x^{(j)}_{\alpha_1})}$$

$$= \text{determinant } (F_{ij})$$

$$F_{2i-1,2j} = -F_{2i,2j-1} = \frac{1}{x_{\alpha_i}-x_{\beta_i}}$$

$$F_{2i-1,2j-1} = F_{2i,2j} = 0$$

Example

$n=4 \quad (x_1 \quad x_2 \quad x_3 \quad x_4)$

$$F_{ij} = \begin{vmatrix} 0 & \frac{1}{x_1-x_2} & 0 & \frac{1}{x_1-x_4} \\ \frac{1}{x_2-x_1} & 0 & \frac{1}{x_2-x_3} & 0 \\ 0 & \frac{1}{x_3-x_2} & 0 & \frac{1}{x_3-x_4} \\ \frac{1}{x_4-x_1} & 0 & \frac{1}{x_4-x_3} & 0 \end{vmatrix}$$

$$\begin{vmatrix} \frac{1}{x_1-x_2} & \frac{1}{x_1-x_4} & 0 & 0 \\ 0 & 0 & \frac{1}{x_2-x_1} & \frac{1}{x_2-x_3} \\ \frac{1}{x_3-x_2} & \frac{1}{x_3-x_4} & 0 & 0 \\ 0 & 0 & \frac{1}{x_4-x_1} & \frac{1}{x_4-x_3} \end{vmatrix} = \begin{vmatrix} \frac{1}{x_1-x_2} & \frac{1}{x_1-x_4} & 0 & 0 \\ \frac{1}{x_3-x_2} & \frac{1}{x_3-x_4} & 0 & 0 \\ 0 & 0 & \frac{1}{x_2-x_1} & \frac{1}{x_2-x_3} \\ 0 & 0 & \frac{1}{x_4-x_1} & \frac{1}{x_4-x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{x_1-x_2} & \frac{1}{x_1-x_4} \\ \frac{1}{x_3-x_2} & \frac{1}{x_3-x_4} \end{vmatrix} \cdot \begin{vmatrix} \frac{1}{x_1-x_2} & \frac{1}{x_3-x_2} \\ \frac{1}{x_1-x_4} & \frac{1}{x_3-x_4} \end{vmatrix} = \begin{vmatrix} \frac{1}{x_1-x_2} & \frac{1}{x_1-x_4} \\ \frac{1}{x_3-x_2} & \frac{1}{x_3-x_4} \end{vmatrix}^2 =$$

$$= \left(\frac{1}{(x_1-x_2)} \cdot \frac{1}{(x_3-x_4)} - \frac{1}{(x_1-x_4)} \cdot \frac{1}{(x_3-x_2)}\right)^2 =$$

$$= \left(\frac{\cancel{x_1x_3} - x_1x_2 - x_3x_4 + \cancel{x_4x_2} - \cancel{x_1x_3} + x_1x_4 + x_2x_3 - \cancel{x_2x_4}}{(x_1-x_2)(x_3-x_2)(x_1-x_4)(x_3-x_4)}\right)^2$$

$$= \left(\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_2)(x_3-x_2)(x_1-x_4)(x_3-x_4)}\right)^2$$

In general

$$F = \left[\det \frac{1}{x_{\alpha_i} - x_{\beta_j}}\right]^2 \Rightarrow$$

$$\det\left(\frac{1}{x_{\alpha_i} - x_{\beta}}\right)_{\alpha\beta} = \text{determinant Cauchy} =$$

$$= \frac{\prod (x_{\alpha_i} - x_{\alpha_j}) \prod (x_{\beta_i} - x_{\beta_j})}{\prod (x_{\alpha_i} - x_{\beta_j})}$$

Total contribution (without $r=0$ cycles)

$$e^{2\pi i \bar{\alpha}(x_{\alpha_1} - x_{\beta_1} + x_{\alpha_2} - x_{\beta_2} \dots + x_{\alpha_r} - x_{\beta_r})} \times \frac{2}{(2\pi i)^n} \cdot (-1)^{n-2r} \times$$

$$\times \left[\frac{\prod_{\substack{i \neq j = 1 \\ i<j}}^{n} (x_{\alpha_i} - x_{\alpha_j}) \prod_{\substack{i \neq j = 1 \\ i<j}}^{n} (x_{\beta_i} - x_{\beta_j})}{\prod_{1 \leq i, j \leq r} (x_{\alpha_i} - x_{\beta_j})}\right]^2$$

$$\times \sum_{\substack{\text{partition of } n-2r \text{ nonphase} \\ \text{variables into } 2r \text{ groups}}} \prod_{i=1}^{r} \prod_{\{\gamma_i\}} \frac{1}{(x_{\alpha_i} - x_{\gamma^i})} \prod_{\{\delta_i\}} \frac{1}{(x_{\delta_i} - x_{\beta_i})}$$

$$[\gamma_1] + [\delta_1] + \dots + [\gamma_r] + [\delta_r] = n - 2r$$

Contribution of $r=0$ cycles

choose arbitrary number of pairs of non-phase variables (k pairs)

The same as above but with $n-2k$

Multiply each by the product of paired variables

$$\frac{1}{2\pi^2(x_i-x_j)^2}.$$

Examples

$n=3$ $\qquad x_1, x_2, x_3 \quad \to \quad r=1$ only ($2r$ groups)

$e^{2\pi i\,(x_1-x_2)}$

$e^{2\pi i\,(x_1-x_3)}$

$e^{2\pi i\,(x_2-x_3)}$

non-phase variables

x_3

x_2

x_1

$e^{2\pi i(x_2-x_1)}$

$e^{2\pi i(x_3-x_4)}$

$e^{2\pi i(x_3-x_1)}$

$$\frac{2^{3-2}}{(2\pi i)^3}(-1)^{3+1}\,\frac{1}{(x_1-x_2)^2}\left[\frac{1}{x_1-x_3}+\frac{1}{x_3-x_2}\right] = -\frac{1}{4\pi^3 i\,(x_1-x_2)(x_2-x_3)(x_3-x_1)}$$

$$R_3^{GUE} = -\frac{1}{2\pi^3(x_1-x_2)(x_2-x_3)(x_3-x_1)}\Big[\sin 2\pi\bar{d}\,(x_1-x_2) + \sin\big(2\pi\bar{d}\,(x_2-x_3)\big) + \sin 2\pi\bar{d}\,(x_3-x_1)\Big]$$

$n = 4. \qquad \underline{x_1, x_2, x_3, x_4}$

$\underline{\text{Phases}} \quad x_1 - x_2, \quad x_1 - x_3, \quad x_1 - x_4, \quad x_2 - x_3, \quad x_2 - x_4, \quad x_3 - x_4,$

$x_1 - x_2 + x_3 - x_4 \qquad x_1 - x_3 + x_2 - x_4 \qquad x_1 - x_4 + x_2 - x_3 + \ldots$

- $\underbrace{x_1 - x_2}_{\text{phase}} \qquad \underbrace{x_3 \; x_4}_{\text{non-phase}}$

$r = 1$

$$\frac{2^{4-2}}{(2\pi i)^4} \underbrace{\frac{1}{(x_1 - x_2)^2}}_{\text{phase}} \Bigg[\underbrace{\overset{(3,4)\;(0)}{\frac{1}{(x_1 - x_3)\,(x_1 - x_4)}}}_{\text{non-phase.}} + \overset{(3)\;(4)}{\frac{1}{(x_1 - x_3)(x_4 - x_2)}} + \overset{(4)\cdot(3)}{\frac{1}{(x_1 - x_4)(x_3 - x_2)}} + \underset{(0)\;(3,4)}{\frac{1}{(x_3 - x_2)(x_4 - x_2)}} \Bigg] \times$$

$$\times \cos 2n\bar{\alpha}(x_1 - x_2)$$

$$= \frac{1}{4\pi^4 (x_1 - x_2)^2} \left[\frac{1}{x_1 - x_3} \left(\frac{1}{x_1 - x_4} + \frac{1}{x_4 - x_2} \right) + \frac{1}{x_3 - x_2} \left(\frac{1}{x_3 - x_4} + \frac{1}{x_4 - x_2} \right) \right] \times$$

$$= \left(\frac{1}{x_1 - x_4} + \frac{1}{x_4 - x_2} \right) \left(\frac{1}{x_1 - x_3} + \frac{1}{x_3 - x_2} \right)$$

$$\frac{(x_1 - x_2)^2}{(x_1 - x_4)(x_4 - x_2)(x_1 - x_3)(x_3 - x_2)}$$

$$= \frac{1}{4\pi^4 (x_1 - x_4)(x_4 - x_2)(x_1 - x_3)(x_3 - x_2)} \cos 2n\bar{\alpha}(x_1 - x_2)$$

$\underline{\text{Total}}:$

$$\frac{1}{2\pi^4 (x_1 - x_2)(x_4 - x_2)(x_1 - x_3)(x_3 - x_2)} \left[\cos 2n\bar{\alpha}(x_1 - x_2) \right] + \text{all permut.}$$

- $x_1 - x_2 + x_3 - x_4 \qquad r = 2$

$$\frac{2^{4-4}}{(2\pi i)^4} \frac{(x_1 - x_2)^2 (x_3 - x_4)^2}{(x_1 - x_3)^2 (x_1 - x_4)^2 (x_2 - x_3)^2 (x_2 - x_4)^2} \cos 2n\bar{\alpha}(x_1 - x_2 + x_3 - x_4) + \ldots$$

- $r = 0, \quad (12)\;(34) \ldots \to \frac{1}{(2\pi^2)^2} \left[\frac{1}{(x_1 - x_2)^2 (x_3 - x_4)^2} + \ldots \right]$ etc.

Riemann zeros calculations

$$R_n^{(\text{Riemann})}(x_1,\dots,x_n) = \frac{(-1)^n}{(2\pi)^n} \sum_{a+b=n} \prod_{i=1}^{a} e^{i x_i \ln p_i} \frac{\ln p_i}{\sqrt{p_i}} \prod_{j=1}^{b} \frac{\ln q_j}{\sqrt{q_j}} e^{-i \bar{x}_j \ln q_j}$$

$$\left\langle \exp\left(iE \ln \frac{p_1 \dots p_a}{q_1 \dots q_b}\right) \right\rangle_E$$

n primes are divided into 2 groups $p_1 \dots p_a$, $q_1 \dots q_b$

$x_1, \dots x_a$, $\bar{x}_1 \dots \bar{x}_b$

x_i are ....

Main contribution: $\boxed{p_1 \dots p_a \approx q_1 \dots q_b}$

(A) none of p's $\neq$ q's

Most general case:

2 groups of primes (p), (q) are split into r blocks

$$\begin{cases} (p)_1 \cdot t_1 = (q)_1 s_1 + k_1 \\ (p)_2 \; t_2 = (q)_2 s_2 + k_2 \\ \dots \\ (p)_r \; t_r = (q)_r s_r + k_r \end{cases}$$

$(p)_i$ and $(q)_i$

$\#(p)_i = a_i$, $\#(q)_i = b_i$

$\sum a_i = a$, $\sum b_i = b$.

t_i and s_i are arbitrary integers s.t. $\prod_{i=1}^{r} t_i = \prod_{j=1}^{r} s_j$

$$\exp\left(iE \ln \frac{(p)_i}{(q)_i}\right) = \exp\left[iE \ln \frac{s_i}{t_i} + iE \frac{k_i}{(q)_i s_i} \right]$$

Type I - off-diagonal contributions (the simplest)

$$P_1 \dots P_a = q_1 \dots q_b + k \qquad (k, p \cdot q)$$

$\ln p_i = w_i \qquad \ln q_j = z_j$

$$\tilde{R}_n^{(a,b)} = \frac{(-1)^n}{(2\pi)^n} \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{a} dw_k \, e^{i x_k w_k} \prod_{j=1}^{b} dz_j \, e^{-i z_j \tilde{x}_j}$$

$$\times \delta\left(\sum_{k=1}^{a} w_k - \sum_{j=1}^{b} z_j \right) \times \left[\sum \left(Z - \sum z_j + \underset{\ln\delta}{\Sigma} \right) \theta\left(\sum z_j - Z - \underset{\ln\delta}{\Sigma} \right) \mu(\delta) \right.$$

inclusion/exclusion $= \delta | (p, q)$

$Z = 2\pi \bar{d}$

general $\quad \delta = \sum_{\alpha} w_\alpha + \sum_{\beta} z_\beta$

α is a sub-set of $1 \dots a$
β is " $1 \dots b$

$\# \alpha = s, \quad \# \beta = t$

$$\approx R_n^{(a,b)} = \frac{(-1)^{\overbrace{n+s+t}^{\text{from } \mu(\delta)}}}{(2\pi)^n} \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^{a} dw_k \, e^{i x_k w_k} \prod^{b} dz_j \, e^{-i z_j \tilde{x}_j}$$

$$\times \left[\int du \, (2\pi \bar{d} - u) . \theta(u - 2\pi\bar{d}) \right] \times$$

$$\times \delta\left(\sum_j z_j - \sum_\beta z_\beta - \sum_\alpha w_\alpha - u \right) . \delta\left(\sum z_j - \sum w_k \right)$$

$1 \dots a = (\alpha) + (\alpha')$

$1 \dots b = (\beta) + (\beta')$

Product of δ-functions $=$ $\delta\left(\sum_{\beta'} z_{\beta'} - \sum_\alpha w_\alpha - u \right) \delta\left(\sum_\beta z_\beta - \sum_{\alpha'} w_{\alpha'} \dots \right.$

(α, β) variables are in inclusion-exclusion here

(α', β') are absent in inclusion/exclusion

Integral = product of 2 integrals J_1 and J_2

add $+i\varepsilon$ add $-i\varepsilon$.

$$J_1 = \int_0^\infty \cdots \int_0^\infty \prod_\alpha dw_\alpha\, e^{i x_\alpha w_\alpha} \prod_{\beta'} dz_{\beta'}\, e^{-i\tilde{x}_{\beta'} z_{\beta'}}$$

$$\times\, \delta\left(u - \sum z_{\beta'} + \sum w_\alpha\right)$$

$$J_2 = \int_0^\infty \cdots \int_0^\infty \prod_{\alpha'} dw_{\alpha'}\, e^{i x_{\alpha'} w_\alpha} \prod_{\beta} dz_\beta\, e^{-i\tilde{x}_\beta z_\beta}$$

$$\times\, \delta\left(u + \sum z_\beta - \sum w_{\alpha'}\right)$$

Calcul of J_1 $\qquad \delta(\;) = \int \frac{d\lambda}{2\pi} e^{-i\lambda(\ldots)}$

$$J_1 = \int_{-\infty}^{+\infty} \frac{d\lambda\, e^{-i\lambda u}}{2\pi} \int_0^\infty \int^\infty e^{-i w_\alpha(\lambda - x_{\alpha'} - i\varepsilon)}\, \mathcal{D} w_\alpha \,.\, e^{i z_{\beta'}(\lambda - \tilde{x}_{\beta'} + i\varepsilon)}\, \mathcal{D} z_{\beta'}$$

$$= \frac{(-1)^{\beta'}}{i^{\alpha+\beta'}} \int_{-\infty}^{+\infty} e^{-i\lambda u} \frac{1}{\prod_\alpha (\lambda - x_\alpha + i\epsilon) \prod_{\beta'} (\lambda - \tilde{x}_{\beta'} + i\varepsilon)}$$

u is always positive $\to \int (z-u)\,\Theta(u-z)\,du$

$\to$ in J_1 one can shift the contour down

$$J_1 = -\frac{(-1)^{'}}{i^{\alpha+\beta'-1}} \sum_{\beta'} e^{-i\tilde{x}_{\beta'} u} \frac{1}{\prod_\alpha (\tilde{x}_{\beta'} - x_\alpha) \prod_{\tilde{\beta}^0 \neq \beta'} (\tilde{x}_{\beta'} - \tilde{x}_{\tilde{\beta}^0})}$$

$$J_2 = \frac{(-1)^\alpha}{i^{\alpha'+\beta-1}} \sum_{\alpha'} e^{i x_{\alpha'} u} \frac{1}{\prod_{\tilde{\alpha}^0 \neq \alpha'} (x_{\alpha'} - x_{\tilde{\alpha}^0}) \prod_\beta (x_{\alpha'} - \tilde{x}_\beta)}$$

Total integral

$$J_1 \cdot J_2 = \frac{(-1)^b}{i^{a+b}} \underbrace{\sum_{\alpha',\beta'} e^{i(x_{\alpha'} - \tilde{x}_{\beta'})u}}_{\text{variables which are not included in inclusion / exclusion}} \times$$

$$\times \prod_{\tilde{\alpha}' \neq \alpha'} \frac{1}{(x_{\alpha'} - x_{\tilde{\alpha}'})} \prod_{\beta} \frac{1}{(x_{\alpha'} - \tilde{x}_{\beta})} \prod_{\alpha} \frac{1}{(\tilde{x}_{\beta'} - x_{\alpha})} \prod_{\tilde{\beta}' \neq \beta'} \frac{1}{(x_{\beta'} - x_{\tilde{\beta}'})}$$

(all α's at the left; all β's at the right)

Total contribution to R_n

$$\frac{(-1)^n}{(2\pi i)^n} \sum_{\alpha',\beta'} e^{i(x_{\alpha'} - \tilde{x}_{\beta'})u} A_{\alpha',\beta'}$$

$$A_{\alpha',\beta'} = \prod_{\tilde{\alpha}' \neq \alpha'} \frac{1}{(x_{\alpha'} - x_{\tilde{\alpha}'})} \prod_{\beta} \frac{1}{(x_{\alpha'} - \tilde{x}_{\beta})} \times$$

$$\prod_{\alpha} \frac{1}{(x_{\alpha} - \tilde{x}_{\beta'})} \prod_{\tilde{\beta}' \neq \beta'} \frac{1}{(\tilde{x}_{\tilde{\beta}'} - \tilde{x}_{\beta'})}$$

$$= \underbrace{\prod_{\substack{k=1 \\ k \neq \alpha'}}^{a} \frac{1}{(x_{\alpha'} - x_k)} \prod_{\substack{j=1 \\ j \neq \beta'}}^{b} \frac{1}{\tilde{x}_j - \tilde{x}_{\beta'}}}_{\text{these factors are the same for all choices of non-phase variables.}} \times \prod_{\alpha} \frac{(x_{\alpha'} - x_{\alpha})}{(x_{\alpha} - \tilde{x}_{\beta'})} \prod_{\beta} \frac{(\tilde{x}_{\beta} - \tilde{x}_{\beta'})}{(x_{\alpha'} - \tilde{x}_{\beta})}$$

Sum over all inclusion / exclusion variables (α) and (β)

$$\sum_{\alpha} \prod_{(\alpha)} \frac{x_{\alpha'} - x_{\alpha}}{x_{\alpha} - \tilde{x}_{\beta'}} = \underset{\substack{\uparrow \\ s=0}}{1} + \sum_{i} \underset{\substack{\uparrow \\ s=1}}{} \frac{x_{\alpha'} - x_i}{x_i - \tilde{x}_{\beta'}}$$

$$+ \underbrace{\sum_{i,j} \frac{x_{\alpha'} - x_i}{x_i - \tilde{x}_{\beta'}} \frac{x_{\alpha'} - x_j}{x_j - \tilde{x}_{\beta'}}}_{s=2} + \dots$$

$$= \prod_{i} \left(1 + \frac{x_{\alpha'} - x_i}{x_i - \tilde{x}_{\beta'}} \right) = \left(x_{\alpha'} - \tilde{x}_{\beta'} \right)^{a} \prod_{i=1}^{a} \frac{1}{(x_i - \tilde{x}_{\beta'})}$$

$$\left(A_{\alpha'\beta'} \right)_{\Sigma_{\alpha,\beta}} = (x_{\alpha'} - \tilde{x}_{\beta'})^{n-2} \prod_{k=1}^{a} \frac{1}{(x_{\alpha'} - x_k)} \frac{1}{(x_k - \tilde{x}_{\beta'})}$$

$$\times \prod_{j=1}^{\ell} \frac{1}{(\tilde{x}_j - \tilde{x}_{\beta'})(x_{\alpha'} - \tilde{x}_j)}$$

All variables are the same.

$$\equiv (x_{\alpha'} - \tilde{x}_{\beta'})^{n-2} \prod_{\substack{\text{all} \\ k=1 \\ k \neq \alpha' \\ k \neq \beta'}}^{n} \frac{1}{(x_{\alpha'} - x_k)(x_k - x_{\beta'})}$$

The result is the same for arbitrary a and b

Eg term $e^{u(x_1-x_2)}$ is the same in

$P = P_2 \quad P_n + k \quad P_1 P_3 \quad P_2 P_4 \quad P_n \; k \quad P_1 P_3 P_4 = P_2 P_5 \quad P_n \; k$

Total 2^{n-2} equal contributions (as 2 variables are fixed

integral of $e^{iux} \to \underline{e^{izx}}$ for $= q_1 \quad + k$

$$R_n^{(I} = \frac{(-1)^n 2^{n-2}}{(2\pi)^n} \sum e^{z(x_\alpha \; x_\beta)} A_{\alpha\beta}$$

$$A_{\alpha\beta} = (x_\alpha - x_\beta)^{n-2} \prod_{k \neq \alpha,\beta} \frac{1}{(x_\alpha - x_k)(x_k - x_\beta)}$$

$$= \frac{1}{(x_\alpha - x_\beta)^2} \sum_{[\gamma]+[\delta]=n-2} \prod_{\{\gamma\}} \frac{1}{x_\alpha \quad x_\gamma} \prod_{\{\delta\}} \frac{1}{x_\delta - x_\beta}$$

General case

$t\,(P_1) = s_1 \; q_1 \; + k$

$t_2\,(P_2) = s_2\,(q_2) + k_2$

$t_r\,(P_r) - s_r\,(q_r) \; k_r$

(P_r and (q_r) are partition of primes

$t \quad s_j =$ integers

$\prod t_i = \prod s_j$

Each contribution exactly as above

The only difference: in J_1 . $u \to u - \ln s + \ln t$

in J_2 $u \to u - \ln t + \ln s$.

and one has to take into account inclusion/exclusion principle for $s = s_1 \dots s_r$ and $t = t_1 \dots t_r$ as for the 4-point correlation function.

Total phase

$$e^{iu(x_\alpha - x_\beta) + i x_\beta \ln s - i x_\alpha \ln t} \sum_{\delta | st} \left(\mu(\delta) e^{i \ln \delta (x_\alpha - x_\beta)} \right)$$

$$= e^{iu(x_\alpha - x_\beta) + i x_\beta \ln s - i x_\alpha \ln t} \times \prod_{\sigma_m | s} \left(1 - e^{i \ln \sigma_m (x_\alpha - x_\beta)}\right) \prod_{\tau_m | t} \left(1 - e^{i \ln \tau_m (x_\alpha - x_\beta)}\right)$$

$$= e^{iu(x_\alpha - x_\beta)} \prod_{\sigma_m | s} \left(e^{i \ln \sigma_m x_\beta} - e^{i \ln \sigma_m x_\alpha} \right) \times \prod_{\tau_m | t} \left(e^{-i \ln \tau_m x_\alpha} - e^{-i \ln \tau_m x_\beta} \right)$$

Total contribution of r groups

$$R_n^{(r)} = \frac{2^{n-2r}\,(-1)^{n+r}}{(2\pi i)^n}\, e^{2\pi i \bar{d}\left[x_{\alpha_1} - x_{\beta_1} + x_{\alpha_2} - x_{\beta_2} + \ldots + x_{\alpha_r} - x_{\beta_r}\right]} A^{\beta_1 \beta_2 \ldots \beta_r}_{\alpha_1, \alpha_2 \ldots \alpha_r}$$

$$A^{\beta_1 \beta_2 \ldots \beta_r}_{\alpha_1 \alpha_2 \ldots \alpha_r} = \underbrace{\sum_{(\gamma)_1 + (\delta)_1 + \ldots + (\gamma)_r + (\delta)_r = n-2r} \prod_{i=1}^{r} \prod_{[\gamma_i]} \prod_{[\delta_i]} \frac{1}{(x_{\alpha_i} - x_{\gamma_i})(x_{\delta_i} - x_{\beta_i})}}_{\text{Sum over all non-phase variables}}$$

$$\times \prod_{i=1}^{r} \frac{1}{(x_{\alpha_i} - x_{\beta_i})^2} \frac{1}{S \cdot t} \prod_{q_m | S} \left(e^{i \ln q_m x_{\beta_i}} - e^{i \ln q_m x_{\alpha_i}}\right)$$

$$\times \prod_{p_m | t} \left(e^{-i \ln p_m x_\alpha} - e^{-i \ln q_m x_\beta}\right)$$

Each factor of S has to be a factor of some t_i

Sum over S_i and t_i is equivalent to the summation over all common factors of S_i and t_j if $i = j$ $m = 1$

$(S_i, t_j) = m$,

$$g_{ij}(m) = \frac{1}{m} \prod_{p | m} \left(e^{i x_{\beta_i} \ln p} - e^{i x_{\alpha_i} \ln p}\right)\left(e^{-i x_{\alpha_j} \ln p} - e^{-i x_{\beta_j} \ln p}\right)$$

$$g_{ij} = \sum_{m=1}^{\infty} \frac{1}{m} g_{ij}(m) \simeq \text{(as for the 4-point function)}$$

$$= \frac{(x_{\alpha_i} - x_{\alpha_j})(x_{\beta_i} - x_{\beta_j})}{(x_{\alpha_i} - x_{\beta_j})(x_{\beta_i} - x_{\alpha_j})}$$

$$\prod_{i\neq j} g_{ij} = \prod_{i<j} g_{ij}^2$$

Final answer

Contribution to R_n from $\pm (p)_i = s_i \; q) + k_i$

$$R_n^{(r)} = \frac{2^{n-2r}(-1)^{n+r}}{(2\pi)^n}\, e^{2\pi i d(x_{\alpha_1} x_{\beta_1} + \dots + x_{\alpha_r} x_{\beta_r})} A^{\beta_1 \dots \beta_r}_{\alpha_1 \dots \alpha_r}$$

$$A^{\beta_1 \dots \beta_r}_{\alpha_1 \dots \alpha_r} = \left[\frac{\prod_{1\le j<i\le r} (x_{\alpha_i} - x_{\alpha_j})(x_{\beta_i} - x_{\beta_j})}{\prod_{1\le i,j\le r} (x_{\alpha_i} - x_{\beta_j})} \right]^2$$

$$\times \sum_{[\gamma]_1 + [\delta]_1 + \dots + [\gamma_r]_r + [\delta]_r = n-2r} \; \prod_{i=1}^{r} \prod_{\{\gamma\}_i} \prod_{\{\delta\}} \frac{1}{(x_{\alpha_i} - x_\gamma^{(i)})(x_\delta^{(i)} - x_{\beta_i})}$$

Each time $p_i = q_j$ it gives $\dfrac{1}{2\pi^2 (x_i - x_j)^2}$

All terms are exactly as for the n-point function of GUE!

Conclusions

- Assuming (very) strong Hardy Littlewood conjectures statistical distribution of Riemann zeros is the same as the one of the Gaussian Unitary Ensemble of random matrices in the limit $E(T) \to \infty$
- Non universal corrections to this limit are calculable

References

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