Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Moments of L-Functions I

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MEAN VALUE THEOREMS AND THE ZEROS OF THE ZETA FUNCTION

OUTLINE

What is a mean value theorem?

II Mean values and zeros.

A sample of important estimates.

IV. Application: A simple zero–density estimate.

V Application: Levinson's method.

Application The number of simple zeros.

I. What is a mean value theorem?

In general it is an estimate for an average of a function. For example

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + Re^{i\theta})|^2 \, d\theta \, .$$

For a function with a Dirichlet series, $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, the average is typically along a vertical segment:

$$\int_0^T |F(\sigma + it)|^2 \, dt \, ,$$

or

$$\int_0^T F(\sigma + it) \, dt \, .$$

Note:

1) The path of integration might not be in the half–plane of convergence.

2) It is customary not to divide by T.

There are many variants, for example, an average over a discrete set of points:

$$\sum_{r=1}^{R} F(\sigma_r + it_r)|^2 \qquad\qquad (\sigma_r + it_r \in \mathbb{C})$$

Example 1. Take $F(s) = \zeta(s)^k$, $\sigma \geq 1/2$, and k a positive integer.

We are interested in the means

$$I_k(\sigma, T) = \int_0^T |\zeta(\sigma + it)^k|^2 \, dt$$

$$\int_0^T |\zeta(\sigma + it)|^{2k} \, dt \, .$$

Example 2. Take $F(s) = (\zeta'(s))^k$, $\mathcal{S}$ a set of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ One can consider the means

$$\sum_{\rho \in \mathcal{S}} |\zeta'(\rho)|^{2k}$$

Example 3. Let

$$F(s) = F_N(s) = \sum_{n=1}^{N} a_n n^{-s}$$

be a Dirichlet "polynomial" of "length" N. One can show that

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{-\sigma-it} \right|^2 dt = (T + O(N \log N)) \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}}$$

This is the "classical mean value theorem for Dirichlet polynomials"

A stronger version, due to H. L. Montgomery and R. C. Vaughan, is

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{-\sigma-it} \right|^2 dt = \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}} (T + O(n))$$

II. Mean values and zeros.

Mean value estimates are used to study the zeros in a variety of ways.

One direct link between them and the zeros of an analytic function is given by

Theorem. (Jensen's Formula) *Let $f(z)$ be analytic for $|z| \leq R$ and suppose that $f(0) \neq 0$. Let $r_1, r_2, \ldots, r_n$ be the moduli of all the zeros of $f(z)$ inside $|z| \leq R$.* ***Then***

$$\log\left(\frac{|f(0)|R^n}{r_1 r_2 \cdots r_n}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta$$

That is, the distribution of zeros of $f(z)$ inside the circle is related to the mean of $\log|f(z)|$ on the circle.

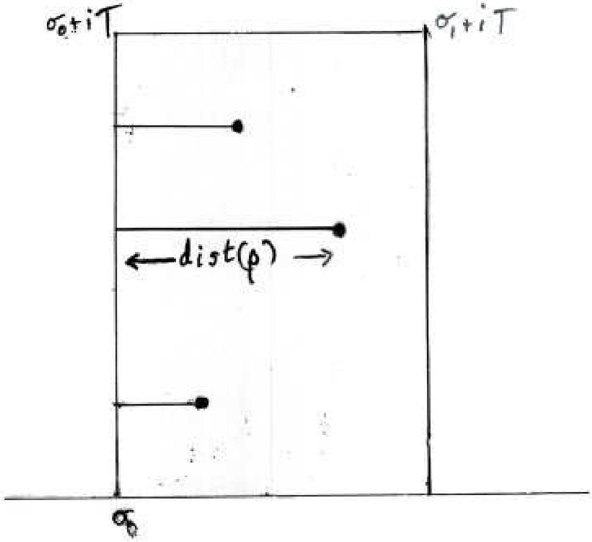
There is an analogous result for rectangles that is more useful when working with Dirichlet series.

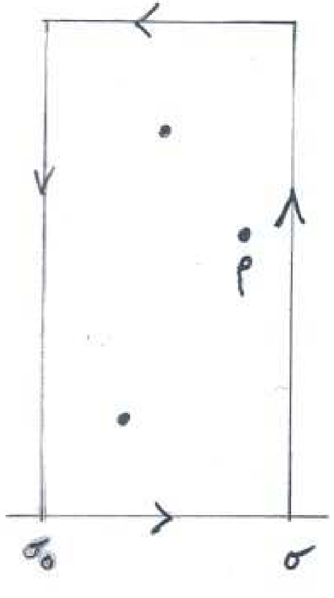
Theorem. (Littlewood's Lemma) *Let $f(s)$ be analytic and nonzero on the rectangle C with vertices σ_0, σ_1, $\sigma_1 + iT$, $\sigma_0 + iT$. Then*

$$2\pi \sum_{\rho \in C} dist(\rho) \quad \int_0^T \log|f(\sigma_0 + it)|\, dt - \int_0^T \log|f(\sigma_1 + it)|\, dt$$

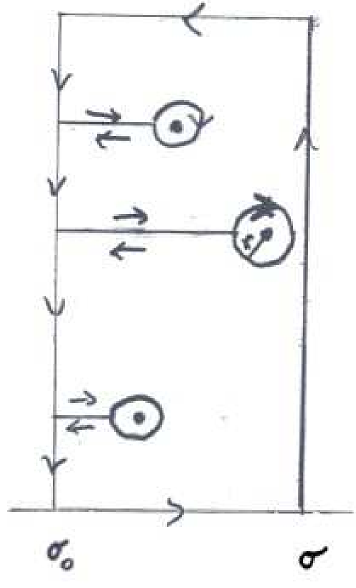
$$+ \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT)\, d\sigma \quad \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\, d\sigma$$

where the sum runs over the zeros ρ of $f(s)$ in C and $dist(\rho)$ is the distance from ρ to the left edge of the rectangle.





C



C'

$\mathcal{L}_\rho$

Proof of Littlewood's Lemma.

Let $\mathcal{C}'$ denote the rectangle $\mathcal{C}$ together with the "loops" $\mathcal{L}_\rho$ around each zero ρ (see the figure). Then

$$\int_{\mathcal{C}} \log f(s)\, ds = \int_{\mathcal{C}'} \log f(s)\, ds + \sum_{\rho \in \mathcal{C}} \int_{\mathcal{L}_\rho} \log f(s)\, ds\,.$$

Now $\log f(s)$ is analytic and single–valued in $\mathcal{C}'$

$$\int_{\mathcal{C}'} \log f(s)\, ds = 0\,.$$

Therefore

$$\int_{\mathcal{C}} \log f(s)\, ds = \sum_{\rho \in \mathcal{C}} \int_{\mathcal{L}_\rho} \log f(s)\, ds\,.$$

If $\rho = \beta + i\gamma$, and the circle in $\mathcal{L}_\rho$ has radius r

$$\int_{\mathcal{L}_\rho} \log f(s)\, ds = \int_{\sigma_0}^{\beta - r} \log f(\sigma + i\gamma \quad)\, d\sigma + \int_0^{2\pi} \log f(re^{i\theta})\, ire^{i\theta} d\theta$$

$$\int_{\sigma_0}^{\beta - r} \log f(\sigma + i\gamma^+)\, d\sigma$$

The second integral $\to 0$ as $r \to 0^+$ The third integral is

$$\int_{\sigma_0}^{\beta-r} \left(\log f(\sigma + i\gamma) + 2\pi i\right) d\sigma .$$

Hence, as $r \to 0^+$

$$\int_{\mathcal{L}_\rho} \log f(s)\, ds \to -2\pi i \int_{\sigma_0}^{\beta} d\sigma = -2\pi i(\beta - \sigma_0) .$$

Therefore

$$\int_{\mathcal{C}} \log f(s)\, ds = -2\pi i \sum_{\rho\in\mathcal{C}} (\beta - \sigma_0) .$$

We may write this as

$$-2\pi i \sum_{\rho\in\mathcal{C}} (\beta - \sigma_0) = \int_0^T \log f(\sigma_1 + it)\, i dt - \int_0^T \log f(\sigma_0 + it)\, i dt$$
$$+ \int_{\sigma_0}^{\sigma_1} \log f(\sigma)\, d\sigma - \int_{\sigma_0}^{\sigma_1} \log f(\sigma + iT)\, d\sigma .$$

The result follows on equating imaginary parts.

Only the first term on the right in Littlewood's Lemma will be signifi-

cant for us, so we will write

$$2\pi \sum_{\rho \in \mathcal{C}} \text{dist}(\rho) = \int_0^T \log|f(\sigma_0 + it)|\,dt - \int_0^T \log|f(\sigma_1 + it)|\,dt$$

$$+ \int_{\sigma_0}^{\sigma_1} \arg f(\sigma + iT)\,d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\,d\sigma$$

as

$$2\pi \sum_{\rho \in \mathcal{C}} \text{dist}(\rho) = \int_0^T \log|f(\sigma_0 + it)|\,dt + \mathcal{E}$$

and ignore $\mathcal{E}$

Usually we cannot estimate the integral directly, so we use the following

trick:

$$\frac{1}{T}\int_0^T \log|f(\sigma_0 + it)|\,dt = \frac{1}{2T}\int_0^T \log(|f(\sigma_0 + it)|^2)\,dt$$

$$\le \frac{1}{2}\log\left(\frac{1}{T}\int_0^T |f(\sigma_0 + it)|^2\,dt\right),$$

("The average of the log is $\le$ the log of the average)

Note: Our mean values appear!

A sample of important estimates.

Recall that

$$I_k(\sigma\ T)\quad \int_0^T |\zeta(\sigma + it)|^{2k}\, dt$$

The case k

$> /2,$

$$I_1(\sigma\ T)\quad \int_0^T |\zeta(\sigma + it)|^2\, dt \quad c_1(\sigma)\, T$$

as $T \to \infty$

Hardy and Littlewood (1918) proved that if $\sigma - /2$ then

$$I_1(1/2\ T)\quad T \log T$$

In particular $|\zeta(\sigma + it)|$ smaller on average for $\sigma > /2$ then for

$\sigma \quad /2$

Since $\zeta(\frac{1}{2} + it)$ has many zeros, we should expect $\zeta(s)$ to be very

erratic on $\sigma = \frac{1}{2}$.

The case $k = 2$.

If $\sigma > 1/2$,

$$I_2(\sigma, T) = \int_0^T |\zeta(\sigma + it)|^4 \, dt \sim c_2(\sigma) T$$

as $T \to \infty$.

Ingham (1926) proved that

$$I_2(1/2, T) \sim \frac{T}{2\pi^2} \log^4 T$$

The case $k > 2$

No asymptotic has yet been proven

Balasubramanian and Ramachandra have shown that

$$I_k(1/2, T) \gg T \log^{k^2} T ,$$

and we expect that

$$I_k(1/2, T) \sim c_k T \log^{k^2} T$$

Conrey and Ghosh conjectured that

$$c_k = \frac{a_k g_k}{\Gamma(k^2+1)}$$

That is, that

$$I_k(1/2,T) \sim \frac{a_k g_k}{\Gamma(k^2+1)} T \log^{k^2} T$$

Here

$$a_k = \prod_p \left(\left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{r=0}^{k-1} \binom{k-1}{r}^2 p^{-r} \right)$$

and g_k is some (then unknown) constant.

J. Keating and N. Snaith used random matrix theory to conjecture the value of g_k for all k. When k is an integer their conjecture is

Conjeture. (Keating–Snaith)

$$g_k = (k^2!) \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Another important mean is

$$(1) \qquad \int_0^T \left|\zeta^{(j)}(\sigma+it)M_N(\sigma+it)\right|^2 dt\,,$$

where

$$M_N(s) = \sum_{1\le n\le N} \frac{\mu(n)}{\cdot}\left(1 - \frac{\log n}{\log N}\right)$$

$M_N(s)$ approximates $1/\zeta(s)$ when $\sigma > 1$. This continues for $\sigma \le 1$ in some sense. Hence,

$$\zeta(s)M_N(s)$$

should be tamer than $\zeta(s)$ on $\sigma = \frac{1}{2}$.

General estimates for means like (1) were proved by Conrey, Ghosh, and Gonek with

$$N = T^{\theta} \qquad \text{and} \qquad \theta < 1/2$$

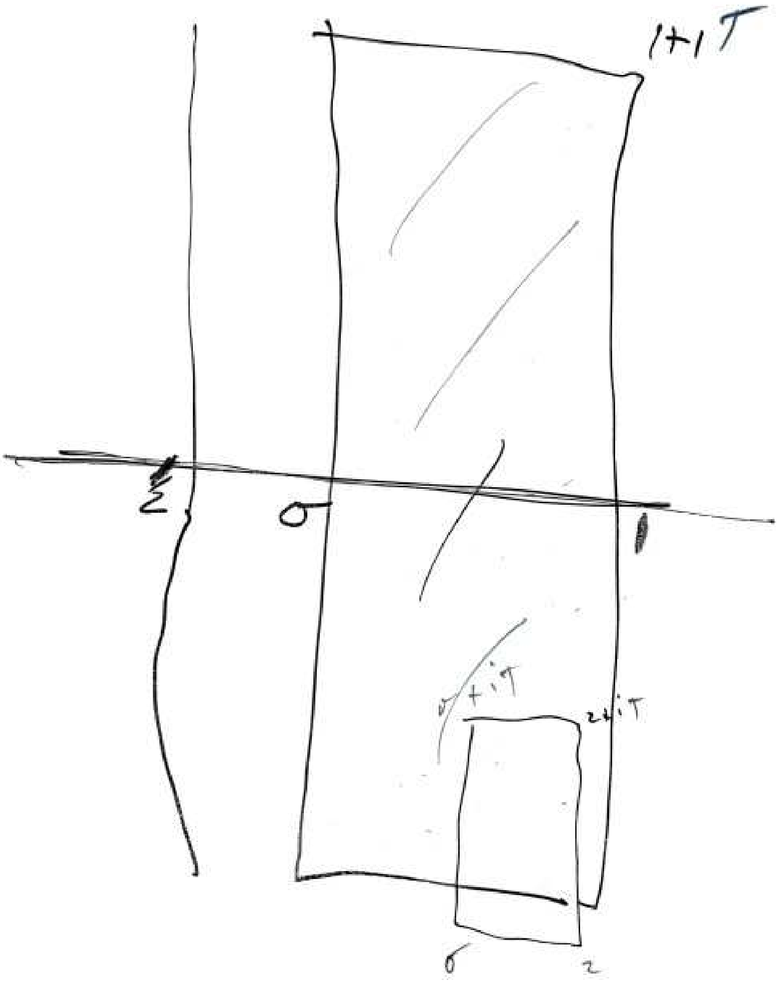
Later, Conrey used Kloosterman sum techniques to show these formulas also hold for $\theta < 4/7$.

Assuming the Riemann Hypothesis and the Generalized Lindelöf Hypothesis are true, Conrey, Ghosh, and Gonek also proved discrete versions of this, including estimates for sums like

$$\sum_{0<\gamma<T} \left|\zeta'(\rho) M_N(\rho)\right|^2$$

Here γ runs over the ordinates of the zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Again

$$N = T^\theta \qquad \text{and} \qquad \theta < 1/2$$

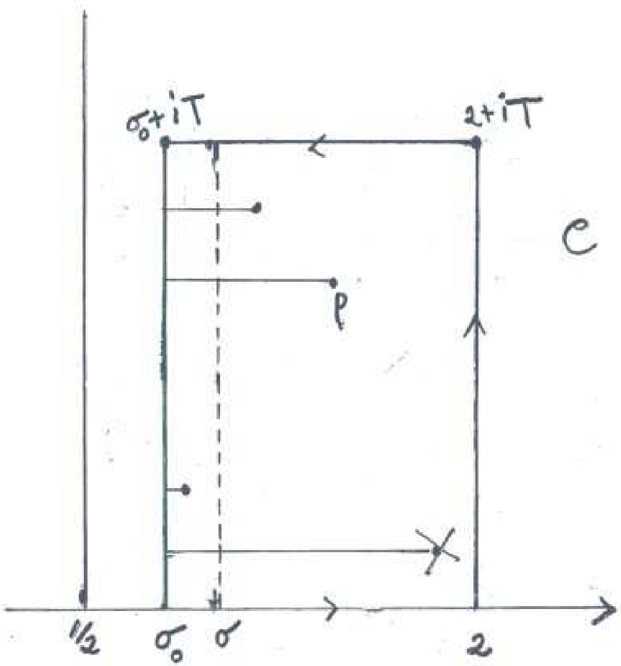

1+iT
σ+iT
2+iT
σ
2

IV. Application: A simple zero–density estimate.

Let

$$N(\sigma,T)=\sum_{\substack{\rho=\beta+i\gamma\\0<\gamma\le T\\\sigma<\beta\le 1}} 1\,.$$

We want an upper bound for $N(\sigma,T)$ when $\frac{1}{2}<\sigma\le 1$ is fixed.



We apply ittlewood' Lemma on the rectangle $\mathcal{C}$ with vertices σ 2

$+ iT$ $\sigma_0 + iT$ where σ_0 fixed

$$\sum_{\rho\in\mathcal{C}} \mathrm{dist}(\rho) - \frac{}{2\pi}\int_0^T \log(|\zeta(\sigma_0 + it)| \; dt + \mathcal{E}$$

et σ be real number with $\sigma_0 < \boldsymbol{\sigma} <$

On the one hand

$$\sum_{\rho\in\mathcal{C}} \mathrm{dist}(\rho) \quad \sum_{\substack{\rho\in\mathcal{C}\\ \sigma<\beta}} \mathrm{dist}(\rho) \gtrsim (\sigma \quad \sigma_0) N(\sigma \; T)$$

On the other hand

$$\frac{}{2\pi}\int_0^T \log(|\zeta(\sigma_0 + it)|)\, dt \quad \frac{}{4\pi}\int_0^T \log(|\zeta(\sigma_0 + it)|^2)\, dt$$
$$< \frac{T}{4\pi} \log(\frac{1}{T}\int_0^T |\zeta(\sigma_0 + it)|^2)\, dt$$

by our trick

Hence

$$(\sigma \quad \sigma_0)N(\sigma \ T) < \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |\zeta(\sigma_0 + it)|^2\right) dt + \mathcal{E}$$

The integral

$$I_1(\sigma_0, T) \quad c_1(\sigma_0)\,T$$

Therefore,

$$(\sigma \quad \sigma_0)N(\sigma \ T) < \frac{T}{4\pi}\log c_1(\sigma_0)$$

and

$$N(\sigma \ T) \ll T$$

Since $N(T) \quad \frac{T}{\ } \log T$ we see that

$$N(\sigma \ T) \ N(T) \quad O\left(\frac{1}{\log T}\right)$$

for any fixed $\sigma > \ /2$.

We may interpret this as saying that only an infinitesimal proportion of the zeros are to the right of any line $Res = \sigma > 1/2$

This was the first zero–density estimate. It was proved by H. Bohr and E. Landau (1914).

Since then, much stronger results have been proven, typically of the form

$$N(\sigma, T) << T^{\lambda(\sigma)}$$

where $\lambda(\sigma) < 1$ and $\lambda(\sigma)$ is decreasing for $\sigma > 1/2$

V. Application: Levinson's method.

Recall that

$$N(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T\}$$

$$\frac{T}{2\pi} \log T$$

and let

$$N_0(T) = \#\left\{\rho = \frac{1}{2} + i\gamma \mid \zeta(\rho) = 0, \quad 0 < \gamma < T\right\}$$

denote the number of zeros *on* the critical line up to height T.

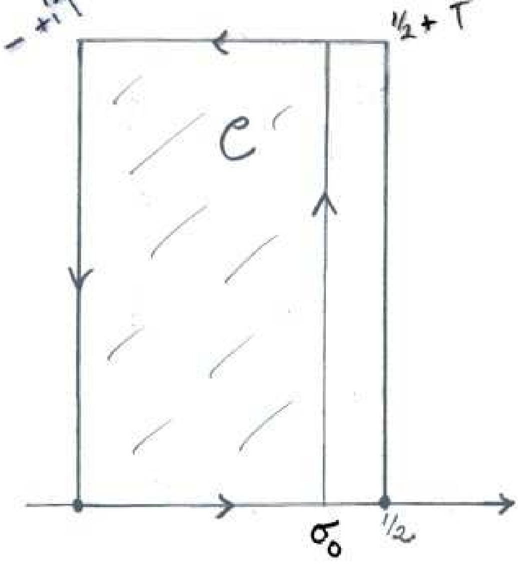
G. H. Hardy (1914) : $N_0(T) \to \infty$ (as $T \to \infty$)

G. H. Hardy-J. E. Littlewood (1921) : $N_0(T) > cT$

A. Selberg (1942) : $N_0(T) > c'N(T)$

N. Levinson (1974) : $N_0(T) > \frac{1}{3}N(T)$

J. B. Conrey (1989) : $N_0(T) > \frac{2}{5}N(T)$



- +iT
1/2 + T
C
σ0
1/2

Levinson's method begins with the following fact first proved by Speiser.

Theorem. (Speiser) *$RH \Longleftrightarrow \zeta'(s) \neq 0$ in $0 < \sigma < \frac{1}{2}$.*

In the early seventies, N. Levinson and H. L. Montgomery proved a quantitative version of this:

Theorem. (Levinson-Montgomery) *$\zeta(s)$ and $\zeta'(s)$ have the same number of zeros inside $\mathcal{C}$ up to $O(\log T)$.*

Proof.

$$\Delta \arg \frac{\zeta'}{\zeta}(s)\Big|_{\mathcal{C}} = O(\log T),$$

and

$$\Delta \arg \frac{\zeta'}{\zeta}(s)\Big|_{\mathcal{C}} = 2\pi(\#\text{ zeros of } \zeta'(s) \text{ in } \mathcal{C} - \#\text{ zeros of } \zeta(s) \text{ in } \mathcal{C}).$$

Sketch of Levinson's method

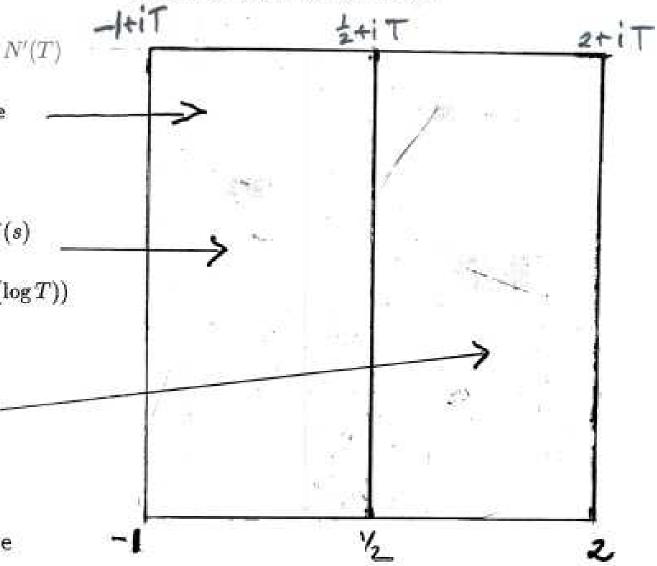
$\zeta'(s)$ has $N'(T)$

zeros here

So does $\zeta(s)$

(up to $O(\log T)$)

and here



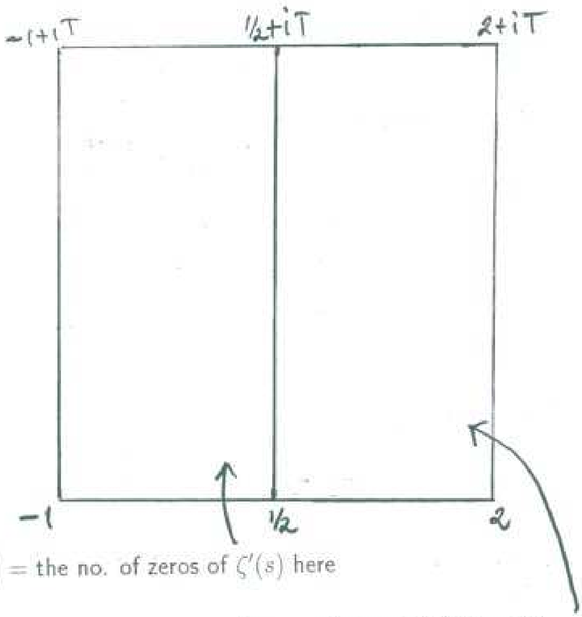
Therefore

$$N(T) \quad N_0(T) + 2N'(T) + O(\log T),$$

or

$$N_0(T) \quad \boldsymbol{N(T)} \quad 2N'(T) + O(\log T).$$

We know $N(T)$, so we need an upper bound for $N'(T)$.



ζ'(1−)

t t to o h t m o

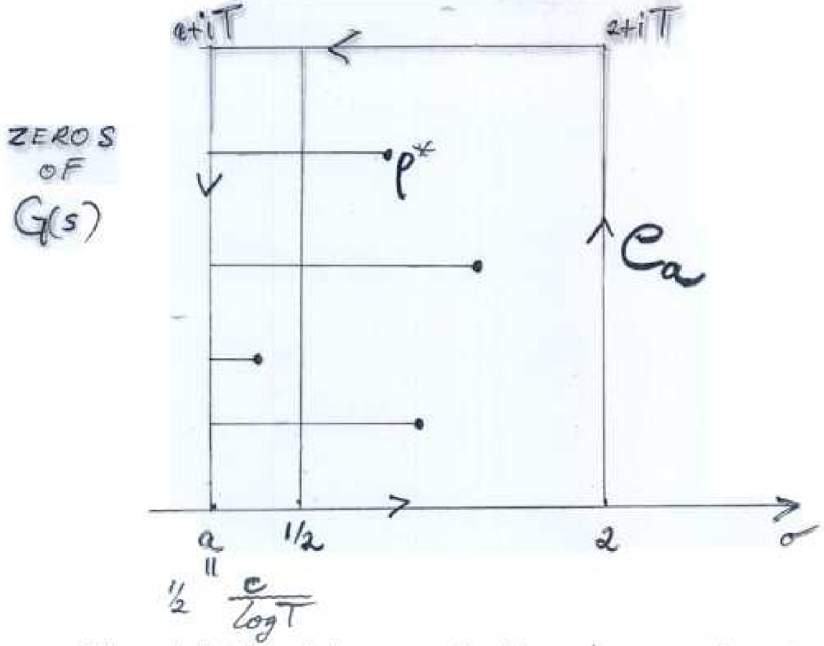
[$< \sigma <$ e $<$ $< T$

$$G \qquad + \frac{\zeta'(s)}{L(s)}$$

w L) $\frac{1}{2}$ o

we d t m o o G n t

ct o t t



We apply Littlewood' Lemma to $\mathcal{C}_a$ with $a = \frac{1}{2} \quad \frac{\ }{\log T}$ and $c > 0$.

$$2\pi \int_0^T \log|G(a+it) \ \mathcal{M}(s) \quad |dt + \mathcal{E}$$

$$\sum_{\substack{zeros\ of\ G\mathcal{M} \\ \in \mathcal{C}_a}} \text{dist}(\rho^*) > \sum_{G \in \mathcal{C}_a} \text{dist}(\rho^*)$$

$$> \sum_{\substack{zeros\ of\ G \\ \in \mathcal{C}_a \\ \beta^* > 1/2}} \text{dist}(\rho^*)$$

$$> \quad (1/2 \quad a)N'(T)$$

Thus,

$$
\begin{aligned}
(1/2 - a)N'(T) &\le \frac{1}{2\pi}\int_0^T \log|GM(a+it)|dt + \mathcal{E} \\
&= \frac{1}{4\pi}\int_0^T \log|GM(a+it)|^2 dt + \mathcal{E} \\
&\le \frac{T}{4\pi}\log\left(\frac{1}{T}\int_0^T |GM(a+it)|^2 dt\right) + \mathcal{E}\,,
\end{aligned}
$$

where

$$
M(s) = \sum_{n \le T^\theta} \frac{a_n}{n^s}\,, \qquad a_n = \mu(n) n^{a-1/2}\left(1 - \frac{\log n}{\log T^\theta}\right),
$$

approximates $1/\zeta(s)$.

Thus, we require an estimate for

$$
\int_0^T |GM(a+it)|^2 dt\,.
$$

This is similar to mean values mentioned before.

Levinson (1974) : One can take $\theta = 1/2 - \epsilon$. This gives

$$N_0(T) > \frac{1}{3} N(T) \qquad (\text{as} T \to \infty) .$$

J B. Conrey (1989) : One can take $\theta = 4/7 - \epsilon$. This gives

$$N_0(T) > \frac{2}{5} N(T) \qquad (\text{as} T \to \infty) .$$

As a function of θ, the asymptotic estimate for

$$\int_0^T |GM(a + it)|^2 dt$$

is the same in both cases.

D. Farmer has argued that this remains true for θ arbitrarily large

Farmer's conjecture imples that

$$N_0(T) \sim N(T) .$$

VI. Application: The number of simple zeros.

Let

$$N_s(T) = \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\, \zeta'(\rho) \neq 0, \quad 0 < \gamma < T\}$$

We believe that for all $T > 0$,

$$N(T) = N_0(T) = N_s(T)\,.$$

H. Montgomery (1974) :

$$RH \Longrightarrow N_s(T) > \frac{2}{3} N(T)$$

via the pair correlation method.

Conrey, Ghosh, Gonek (1999):

$$RH + GLH \Longrightarrow N_s(T) > \frac{19}{27} N(T) = (.703\ldots) N(T)\,.$$

Sketch of the method

This time we use discrete mean values. By the Cauchy–Schwarz inequality

$$\Big|\sum_{0<\gamma<T} \zeta'(1/2+i\gamma)\Big|^2 \le \Big(\sum_{\substack{0<\gamma\le T\\ 1/2+i\gamma\ is\ simple}} 1\Big)\Big(\sum_{0<\gamma<T} |\zeta'(1/2+i\gamma)|^2\Big)$$

An asymptotic estimate for the means provides a lower bound for $N_s(T)$. But this only leads to $N_s(T) > cT$.

We lose in applying the Cauchy–Schwarz inequality. To minimize the loss we mollify $\zeta'(1/2+i\gamma)$ by a Dirichlet polynomial $M_N(s)$

$$\Big|\sum_{0<\gamma<T} \zeta'(\rho)M_N(\rho)\Big|^2$$

$$\le \Big(\sum_{\substack{0<\gamma\le T\\ \rho=1/2+i\gamma\\ is\ simple}} 1\Big)\Big(\sum_{0<\gamma<T} |\zeta'(\rho)M_N(\rho)|^2\Big).$$

An elaboration of the method shows that on the same hypotheses at least 95.5% of the zeros of $\zeta(s)$ are either simple or double.