Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Toeplitz determinants & connections to random matrices I

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Lecture 1

Consider a sequence of complex numbers

$$\{ a_i \}_{i= \ \infty}$$

and the $n \times n$ matrix

$$T_n = (a_{i,j})_{i,j=}^{n}$$

T_n looks like

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-n+1} \\ a_1 & a_0 & a_{-1} & a_{-n+2} \\ a_2 & a_1 & a_0 & \\ \vdots & & \ddots & \\ & & & \ddots \\ a_{n-1} & \cdots & & a_0 \end{pmatrix}$$

T_n is called a finite Toeplitz matrix

Basic question is what happens to

D_n $\det T_n$ as $n \to \infty$

The finite matrix looks like a "truncation" of the infinite one

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdot \\ a_1 & a_0 & a_{-1} & \\ a_2 & a_1 & a_0 & \\ & & & \end{pmatrix}$$

Can we get information about determinants from the infinite array?

We introduce the Hardy space H^2.

$$H^2 = \{ f \in L^2(S^1) \,/\, f_k = 0,\ k < 0 \}$$

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta}\, d\theta$$

$$f \sim \sum_0^{\infty} f_k e^{ik\theta}$$

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})}\, d\theta$$

H^2 is a closed subspace of L^2. We denote the orthogonal projection of L^2 onto H^2 by P.

We now let $\varphi \in L^\infty(S^1)$ and define

$$T(\varphi) \quad H^2 \longrightarrow H^2 \quad \text{by}$$

$$T(\varphi)f = P(\varphi \cdot f)$$

$T(\varphi)$ is called a Toeplitz operator with symbol φ

Let $e_k = e^{ik\theta}$ Then

$$\langle T(\varphi)e_k, e_j \rangle = \langle P(\varphi \cdot e_k), e_j \rangle$$

$$= \langle \varphi e_k, P e_j \rangle = \langle \varphi e_k, e_j \rangle$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{i\theta})\, e^{ik\theta} e^{-ij\theta}\, d\theta$$

$$= \varphi_{j-k}$$

We need one more operator $H(\varphi): H^2 \longrightarrow H^2$.

$$H(\varphi)f = P(\varphi Jf)$$

where

$$J(f)(e^{i\theta}) = \frac{1}{e^{i\theta}} f(e^{-i\theta})$$

$$\tilde{f}(e^{i\theta}) = f(e^{-i\theta}) = \sum_{-\infty}^{\infty} f_k e^{-ik\theta}$$

$$\langle H(\varphi)e_k, e_j\rangle = \langle P(\varphi J(e_k)), e_j\rangle$$

$$= \langle \varphi e_k \;\;, e_j\rangle$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(e\;\theta)\, e^{-i(k+\;+j)\theta}\, d\theta$$

$H(\varphi)$ looks like

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \\ \varphi_2 & \varphi_3 & & \\ \varphi_3 & & & // \\ & & & \end{pmatrix}$$

Facts about Toeplitz Operators.

$T(\varphi)$ is bounded.

$$\|T(\varphi)f\|_2 = \|P(\varphi f)\|_2$$
$$\leq \|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2$$

If $\psi_+ \in L_\infty \cap H^2$, then

$$T(\varphi\psi_+) = T(\varphi)T(\psi_+).$$

follows since

$$P(\varphi\psi_+ f) = P(\varphi P(\psi_+ f)).$$

$$T(\varphi)^* = T(\bar{\varphi}).$$

If $\overline{\psi_-} \in H^2$, then

$$T(\psi_-\varphi) = T(\psi_-)T(\varphi).$$

5 $$T(\varphi\psi) = T(\varphi)T(\psi) + H(\varphi)H(\tilde{\psi})$$

$$\tilde{\psi}(e^{i\theta}) = \psi(e^{-i\theta})$$

Last formula follows since

$$(\varphi\cdot\psi)_k = \sum_{-\infty}^{\infty}{}' \varphi_\ell \psi_{k-\ell}$$

Finally, define

$$P_n : H^2 \longrightarrow H^2$$

$$P_n(f_0, f_1, \ldots, f_k, \ldots)$$

$$= (f_0, f_1, \ldots, f_{n-1}, 0, 0, \ldots)$$

We think of $T_n(\varphi)$

as $\quad P_n T(\varphi) P_n$

Determinants

If M is a finite matrix,

$$\det M \quad \prod_{i=1}^{n} \beta_i$$

β_i eigenvalues

What about $\prod_{i=1}^{\infty} \beta_i$?

Let $\beta_i = 1 + \lambda$

so $\det(I + K) \quad \prod_i^{\infty}(1+\lambda_i)$

makes sense if $\sum |\lambda_i| < \infty$

This leads us to the idea of trace class operators

K is trace class

if $\|K\|_1 = \sum_{n=1}^{\infty} \langle (KK^*)^{1/2} e_n, e_n \rangle < \infty$

(If K is trace class, then

it is compact and

$\sum' |\lambda_i| < \infty$)

K is called Hilbert-Schmidt

if $\sum'_{i,j} |\langle K e_i, e_j \rangle|^2 < \infty$

If K and K_2 are

Hilbert Schmidt operators

then $K K_2$ is

trace class

Facts about Trace Class Operators.

Trace class operators form an ideal in the set of bounded operators

. $\det P_n (I + K) P_n$

$\longrightarrow \det (I + K)$

if K is trace class

3. If $A_n \to A$, $B_n^* \longrightarrow B^*$ strongly (pointwise) then

$A_n K B_n \longrightarrow AKB$ in the trace norm.

How does one compute $\det(I + K)$?

Two ways

If $I + K \quad e^{A}$

then $\det e^{A} = e^{\operatorname{tr} A}$

(A trace class)

Suppose $K_1 K_2 - K_2 K_1$

is trace class, then

$\det\left(e^{K_1} e^{K_2} e^{K_1} e^{-K_2}\right)$

$= e^{\operatorname{trace}(K \; K_2 \quad K_2 K_1)}$

Lemma 1 Suppose $\varphi \in L^\infty$ satisfies $\sum_{-\infty}^{\infty}{}' |k| |\varphi_k|^2 < \infty$

Then $H(\varphi)$ and $H(\tilde{\varphi})$ are both Hilbert Schmidt.

Pf $H(\varphi)$ has matrix form

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \\ \varphi_2 & \varphi_3 & & \\ \varphi_3 & & & \\ & & & \end{pmatrix}$$

so

$$\sum_{i,j} |\langle H(\varphi) e_j, e_k \rangle|^2 = \sum_{1}^{\infty} k |\varphi_k|^2$$

is finite

Cor. If φ, ψ satisfy the conditions of Lemma 1, then

$$H(\varphi)H(\tilde{\psi})$$

is trace class and

$$\det(I + H(\varphi)H(\tilde{\psi}))$$

is well-defined.

Question - Is

$$T(\varphi) = I + \text{trace class}$$

NO, But almost —

Theorem 1

Suppose $\varphi = \varphi_- \varphi_+$

where $\varphi_+, \bar{\varphi}_- \in H^2$

and $\log \varphi_\pm$ satisfy

$$\sum_{-\infty}^{\infty} |(\log \varphi_\pm)_k| + \sum_{-\infty}^{\infty} |k| |(\log \varphi_\pm)_k|^2 < \infty$$

Then $T(\varphi) T(\varphi^{-1}) = I + K$

K trace class and

$$\det(T(\varphi) T(\varphi^{-1}))$$

$$= \exp\left(\sum_1^{\infty} k s_k s_{-k} \right)$$

where $(\log \varphi)_k = s_k$.

Pf Functions which satisfy the condition

$$\sum |f_k| + \sum |k|\,|f_k|^2 < \infty$$

form a Banach algebra so if $\log \varphi_\pm$ satisfy this, so do $\varphi_\pm$, $\frac{1}{\varphi_\pm}$, φ, φ^{-1}

Thus $T(\varphi)T(\varphi^{-1})$ $I + H(\varphi)H($

Each of last two are Hilbert-Schmidt

$$\det T(\varphi)\,T(\varphi^{-1})$$

$$= \det T(\varphi_-)T(\varphi_+)T(\varphi_-^{-1})T(\varphi_+^{-1})$$

$$= \det\left(e^{K_1} e^{K_2} e^{-K_1} e^{-K_2}\right)$$

where $K_1 = T\log(\varphi_-)$

$K_2 = T\log(\varphi_+)$

So

$$\det\left(T(\varphi)\,T(\varphi^{-1})\right)$$

$$= \exp\left\{\operatorname{trace}\left[T(\log\varphi_-),\, T(\log\varphi_+)\right]\right\}$$

$$T(\log\varphi_-)\,T(\log\varphi_+) - T(\log\varphi_+)\,T(\log\varphi_-)$$

$$= H(\log\varphi_+)\,H(\log\tilde{\varphi}_-)$$

$$(\log\varphi)_k = \begin{cases} (\log\varphi_+)_k & k \geq 0 \\ (\log\varphi_-)_k & k < 0 \end{cases}$$

and $\operatorname{trace} H(\log\varphi_+)\,H(\log\tilde{\varphi}_-)$

becomes $\sum_{1}^{\infty} k S_k S_{-k}$

or $\det T(\varphi) T(\varphi^{-1})$

$$= \exp\left(\sum_{1}^{\infty} k S_k S_{-k}\right)$$

We are computing the trace of

$$\begin{pmatrix} S_1 & S_2 & S_3 & \cdots \\ S_2 & S_3 & & \\ S_3 & & & \\ & & & \end{pmatrix} \begin{pmatrix} S_{-1} & S_{-2} & S_{-3} & \\ S_{-2} & S_{-3} & & \\ S_{-3} & & & \\ & & & \end{pmatrix}$$

Cor is the Strong Szegö Limit Th

$$D_n(\varphi) / G(\varphi)^n \longrightarrow \exp\left(\sum_{1}^{\infty} k\, s_k\, s_{-k} \right)$$