Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Spacing distributions for random matrix ensembles I

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Spacing distributions

Motivation

A theoretical prediction from the late '50's

highly excited energy levels of complex nuclei $\overset{\text{distr.}}{=}$ eigenvalues of large GOE matrices

$n + {}^{166}Er$

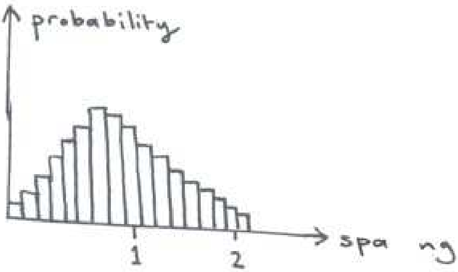
Q./ How can this hypothesis be tested?

A./ Compare computable statistical quantities e g Sp c ng d str but n

L st numer ie $\left(\begin{array}{l} \text{f id} \\ \text{ique s} \\ \text{th it } \langle \text{spacing} \rangle = 1 \end{array} \right)$

f a pa ngs

C st uct h togram



pr b b ty $\dfrac{\#\text{ spacings in each range (b)}}{\text{total } \#\text{ spacings}}$

Q F r ge GOE m t c can th s c mput d n yt ca y?

Wigner surmise $\left(\begin{array}{l}\text{approximate}\\ \text{analytic form}\end{array}\right)$

Strategy

① Start with joint eigenvalue p.d.f. of 2x2 GOE matrices

$$\frac{1}{G_2}(\lambda_2-\lambda_1)e^{-(c\lambda_1^2+c\lambda_2^2)/2} \quad , \quad \lambda_2>\lambda_1$$

$2\sqrt{\pi}/c^{3/2}$

variances

Chosen to 'unfold' spectrum

② Calculate the averaged spacing p.d.f.

Wigner

$$p^W(s) = \int_{-\infty}^{\infty} p(x, x+s)\,dx$$

distance between eigenvalues $= s$

$$= \frac{1}{G_2} s e^{-cs^2/4} \int_{-\infty}^{\infty} e^{-cx^2-csx}\,dx$$

$$= \left(\frac{\pi}{c}\right)^{1/2}\frac{1}{G_2} s e^{-cs^2/4} = \frac{c}{2} s e^{-cs^2/4}$$

③ Cho se c so th t

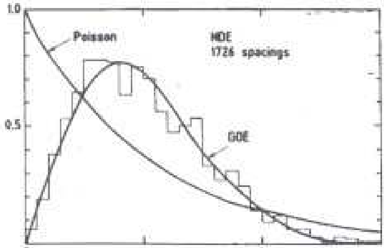
$$\int^{\infty} s p^W(s) \, ds \quad 1$$

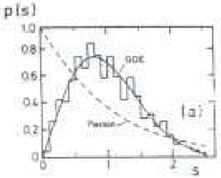
F d c π

$$\Rightarrow \quad p(s) \quad \frac{\pi s}{2} e^{\pi s^2 4}$$

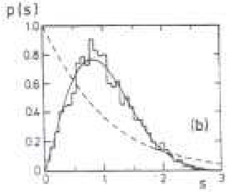
C pa e w th histogram of dat. from GOE matrices ✓

Compare GOE pred ction (Wigner sur se) aga nst clear spectrum data

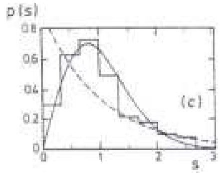




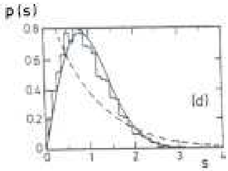
Sinai billiard



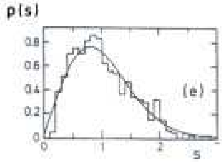
Hydrogen atom in a strong mag. field.



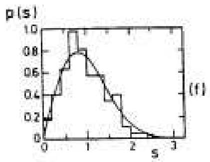
NO_2 molecule



Vibrating quartz block shaped like a 3d Sinai billiard



Microwave spectrum of 3d cavity

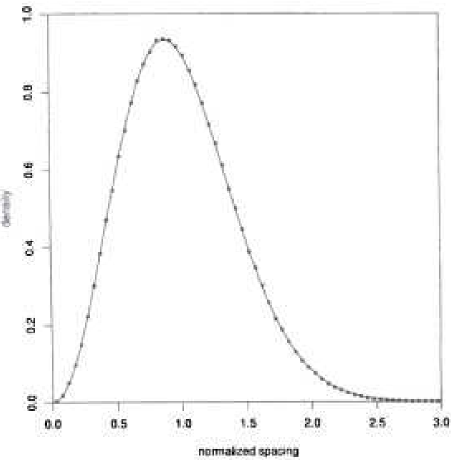


Vibrating elastic disk shaped like a quater stadium.

A theoretical prediction from the 70's & 80's (Montgomery-Odlyzko "law")

large zeros of Riemann zeta on critical line $\overset{\text{distr.}}{=}$ eigenvalues of large GUE matrices

- To test, compare spacing distributions



over a million zeros near 2×10^{20}

- precise determination of $p_{sd}(s)$ for zeta f^{n} zeros requires precise determination of $p_{sd}(s)$ for large GUE matrices

- Wigner surmise is not precise and must be replaced by the "exact" value of $p_{sd}(s)$.

Theory relating to $p_{sd}(s)$

In general (for a translationally invariant state — bulk)

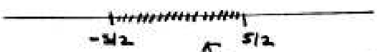
$$p_{sd}(s) = \frac{d^2}{ds^2} \underbrace{E^{\text{bulk}}(0;s)}$$

prob. an interval of length s contains no eigenvalues (gap probability)

$$E^{\text{bulk}}(0;s) = \lim_{N\to\infty} \int_{I} d\lambda_1 \cdots \int_{I} d\lambda_N \, \underbrace{p(\lambda_1,\ldots,\lambda_N)}$$

(scaled) prob. density f^{n} for the eigenvalues

$(-\infty,\infty) \setminus (-s/2, s/2)$



excluded region

Task compute E^{bulk} o s) n the cases

f $\begin{cases} \text{orthogonal} \\ \text{unitary} \\ \text{symplectic} \end{cases}$ symmetry

Some <u>general formulas</u>

Consider the gener ting f

E_N a_1, a_2 ξ) $\xi\int^{a_2} dx_1$ $\xi\int$ $)dx$

$p(x_1$ $, x_N$

E_N a_2 ξ) $\underbrace{E_N(0}_{\text{prob.}}$ a a_2)

eigenvalues in

a_1 a_2)

$(1)^n$ ∂

n $\partial\xi^n$ E_N a $\xi)\Big|_{\xi\ 1}$ $-$ $\underbrace{E_N\ n,\ a_1, a_2}_{\text{prob}}$)

eigenvalue

(a a

Introduce the k-point distribution f$^{\underline{n}}$

$$\rho_{(k)}(x_1,\dots,x_k) := \frac{N!}{(N-k)!}\int_{-\infty}^{\infty} dx_{k+1} \int_{-\infty}^{\infty} dx_N\, \rho(x_1,\dots,x_N)$$

Expanding $E_N((a_1,a_2);\xi)$ in a power series in ξ shows

$$E_N((a_1,a_2);\xi) = 1 + \sum_{k=1}^{N} \frac{(-\xi)^k}{k!} \int_{a_1}^{a_2} dx_1 \cdots \int_{a_1}^{a_2} dx_k\, \rho_{(k)}(x_1,\dots,x_k)$$

Technical point

We must unfold the spectrum

$$x_\ell \mapsto \underbrace{\alpha_N}_{\text{change of origin}} + \underbrace{\beta_N}_{\text{change of scale}} x_\ell =: S_N(x_\ell)$$

The formal structure of the expansion remains

$$E^{\text{scaled}}((a_1,a_2);\xi) := \lim_{N\to\infty} E_N((S_N(a_1), S_N(a_2));\xi)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-\xi)^k}{k!} \int_{a_1}^{a_2} dx_1 \int_{a_1}^{a_2} dx_k\, \rho^{\text{bulk}}_{(k)}(x_1,\dots,x_k)$$

The case of unita y symmetry

$$\rho_k^{bulk} \; x_1 \; .,x_k) \quad \det\left[K \; x_j \; x_{k'}) \right]_{k \quad k}$$

$$K(x \; y \qquad \frac{s \; n\, \pi (x \; y)}{\pi \; x \; y)}$$

But general

$$1 + \sum_{k=1} \frac{(-\xi)^k}{k} \;{}^{a_2} dx \quad \int_{a_1}^{a_2} dx_k \; \det\left[K \quad {}_j \; x_{k'} \right]_{k=1,\dots,k}$$

$$\det(1 \quad \xi K) \qquad \Pi \quad 1 \; \xi \lambda \;)$$

integra operator

$a_1, a_2)$ with kernel

$K(\;\; y)$

eigenvalues

of K

$$K[f] \qquad \int_a^{a_2} K \quad x \; y \;\; f \; y)\, dy$$

Exact evaluation of the spacing distribution:

Historical development

1950's — Wigner makes guess

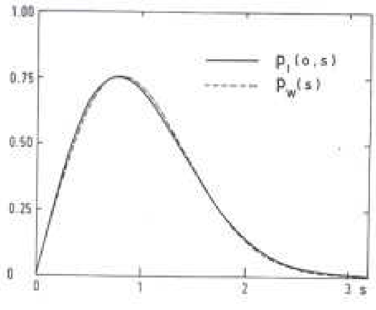
$$p_{sd}(s) \propto s^{\beta} e^{-cs^2}$$

1961 — Gaudin gives formula for $E_1(0;s)$ in terms of Fredholm determinant:

$$E_1(0;s) = \det(1 - K_1)$$

$$K_1 f[x] := \int_{-s/2}^{s/2} \frac{1}{2}\left(\frac{\sin\pi(x-y)}{\pi(x-y)} + \frac{\sin\pi(x+y)}{\pi(x+y)}\right) f(y)\,dy$$

Numerical evaluation shows (small) discrepancy with Wigner surmise



1961 — Mehta

$$E_1(0;(-t,t);e^{-x^2/2};N) = E_2(0;(0,t^2);\underbrace{y^{-1/2}e^{-y}}_{y>0};N/2)$$

Dyson

1962 $E_2\ 0\ s) - \det(1 + K_2)$

$K_2 f[x] \quad \int_{-s/2}^{s/2} \frac{\sin\pi(x-y)}{\pi(x-y)} f\ y\ dy$

962 I ter re ationsh p

$E_2\ 0\ s) \quad E_1(0\ s\ \Big(E\ 0\ s\ + \underbrace{E_1\ 1, s}\Big)$

prob f 1
eigenvalue n interva
of length s $\beta = 1$

Th (Dyson Gunson

$a\ t(COE_N \cup COE_N)$

CUE_N

integrate t every second
eigenv lu.

1963 Th (Dyson & Mehta)

$a\, t(\mathrm{COE}_{2N}) \quad \mathrm{CSE}_N$

integrate out every second eigenva ue

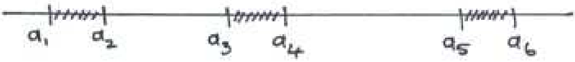
$\Rightarrow E_{4.N}(0\ s) \quad E_{1\ 2N}(0\ s) + \frac{1}{2} E_{1,2N}\ 1\ s$

$\Rightarrow E_4\ 0\ s) \quad E_1(0 \cdot 2s) + \frac{1}{2} E_1\ 1;2s)$

1980 Jimbo, Miwa, Mori & Sato

Relate $E_2\left(0; \bigcup_{j=1}^{p} (a_{2j-1}, a_{2j})\right)$

to integrable system theory



Also, prove

$$\boxed{E_2(0;s) = \exp \int_0^{\pi s} \frac{\sigma(u)}{u}\, du}$$

where $\sigma(u)$ satisfies

$$(u\sigma'')^2 + 4\left(u\sigma' - \sigma + (\sigma')^2\right) = 0$$

$$\sigma(u) \underset{u \to 0}{\sim} -\frac{u}{\pi}$$

Painlevé V eq$^{\underline{n}}$
(in σ-form)

$\Rightarrow$ (Gaudin formula)

$$\boxed{E_1(0;s) = \left(E_2(0;s)\right)^{1/2} \exp\left(\frac{1}{2}\int_0^{\pi s} \left(-\frac{d}{dx}\frac{\sigma(x)}{x}\right)^{1/2} dx\right)}$$

Knowledge of E_1 & E_2 $\Rightarrow$ E_4:

$$E_4(0;s/2) = \frac{1}{2}\left(E_1(0;s) + \frac{E_2(0;s)}{E_1(0;s)}\right)$$

$$\Rightarrow \boxed{E_4(0;s/2) = \left(E_2(0;s)\right)^{1/2} \cosh\left(\frac{1}{2}\int_0^{\pi s} \left(-\frac{d}{dx}\frac{\sigma(x)}{x}\right)^{1/2} dx\right)}$$

1989 Korepin, Its & Izergin identify
$$\det(1-K),$$
for K with kernel
$$K(x,y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x-y}, \qquad (*)$$
with integrable system theory

1991 Mehta, then Dyson, simplify JMMS derivation of Painlevé evaluation of $E_2(0;s$

1992 1994 Tracy and Widom

- Give formalism to evaluate $\det(1+K)$ in terms of Painlevé transcendents for other K of the form $(*)$ and
$$m(x)\phi'(x) = A(x)\phi(x) + B(x)\psi(x)$$
$$m(x)\psi'(x) = -C(x)\phi(x) - A(x)\psi(x)$$
m A, B, C polys

- Apply formalism for various K of interest in random matrix theory

1995 Adler & van Moerbeke

introduce methods based on

Virasora operators

$$= \sum_{k=0}^{p} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{p-k}} + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{p+k}} , \frac{\partial}{\partial t_0} = N$$

$$L_p, L_q] = (p-q) L_{p+q}$$

and the KP eq$^{\underline{n}}$

- generating f$^{\underline{n}}$

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 + 3\left(\frac{\partial}{\partial t_1}\right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log Z_N$$

$$+ 6 \left(\frac{\partial^2}{\partial t_1^2} \log Z_N \right)^2 = 0$$

2000 F. & Witte use Okamoto τ-f$^{\underline{n}}$

theory of Hamiltonian formulation

of the Painlevé equations

$$\dot{q} = \frac{\partial H}{\partial p} , \quad \dot{p} = \frac{\partial H}{\partial q}$$

$$H = \frac{d}{dt} \log \tau$$

2001 Borodin & Deift make use o

Riemann - Hilbert approach to

Schlesinger eq$^{\underline{ns}}$

Tracy - Widom theory

Objective   To characterize

$$E^{soft}((s,\infty);\xi) = \det(1 - \xi K^{soft}_{(s,\infty)})$$

where $K^{soft}_{(s,\infty)}$ is the integral operator on $(s,\infty$ with kernel

$$K^{soft}(x,y) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x \quad y}$$

as the solution of a differential eq^n

Result

$$E^{soft}((s,\infty);\xi) = \exp\left(-\int_s^\infty (t-s)\, q^2(t;\xi)\, dt\right)$$

where q satisfies the particular Painlevé II eq^n

$$q'' = tq + 2q^3 \qquad q(t;\xi) \underset{t\to\infty}{\sim} \xi Ai(\xi)$$

Ingredients

$R(x,y)$ – kernel of the resolvent operator

$$\xi K_J (1 - \xi K_J)^{-1} =: R_J$$

$\rho(x,y)$ – kernel of $(1 - \xi K_J)^{-1}$

Introduce structured form of the kernel

$$\xi K(x,y) = \frac{\phi(x)\psi(y) - \phi(y)\psi(x)}{x - y} \qquad \begin{cases} \phi(x) = \sqrt{\xi}\,Ai(x) \\ \psi(y) = \sqrt{\xi}\,Ai'(y) \end{cases}$$

Now define (with $J = (a_1, a_2) = (s, \infty)$)

$$Q(x) := (1 - \xi K_J)^{-1}\phi = \int_{a_1}^{a_2} \rho(x,y)\phi(y)\,dy$$

$$q_j = Q(a_j) := \lim_{\substack{x \to a_j \\ x \in (a_1, a_2)}} Q(x)$$

$$P(x) := (1 - \xi K_J)^{-1}\psi = \int_{a_1}^{a_2} \rho(x,y)\psi(y)\,dy$$

$$p_j = P(a_j) := \lim_{\substack{x \to a_j \\ x \in (a_1, a_2)}} P(x)$$

$$u = \int_{a_1}^{a_2} \phi(y) Q(y)\,dy$$

$$v = \int_{a_1}^{a_2} \phi(y) P(y)\,dy = \int_{a_1}^{a_2} \psi(y) Q(y)\,dy$$

Now derive <u>universal</u> eq

(i) $R(a_j, a_k) = \dfrac{q_j p_k - p_j q_k}{a_j - a_k}$ $\qquad j \neq k)$

(ii) $\dfrac{\partial}{\partial a_j} \log \det(1 - \xi K_J) = (-1)^{j-1} R(a_j, a_j)$

(iii) $\dfrac{\partial q_j}{\partial a_k} = (-1)^k R(a_j, a_k) q_k \qquad (j \neq k)$

(iv) $\dfrac{\partial p_j}{\partial a_k} = (-1)^k R(a_j, a_k) p_k \qquad (j \neq k)$

(v) $\dfrac{\partial u}{\partial a_k} = (-1)^k q_k^2$

(vi) $\dfrac{\partial v}{\partial a_k} = (-1)^k p_k q_k$

e.g. (ii)

$$\frac{\partial}{\partial a_j} \log \det(1 - \xi K_J) = \frac{\partial}{\partial a_j} \operatorname{Tr} \log(1 - \xi K_J)$$

$$= \operatorname{Tr}\Big(-\xi (1 - \xi K_J)^{-1} \underbrace{\frac{\partial}{\partial a_j} K_J}_{(-1)^j K_J \delta(a_j - y)}\Big)$$

$$(-1)^{j-1} \operatorname{Tr}\left(R_J \delta(a_j - y)\right)$$

$$= (-1)^{j-1} R(a_j, a_j)$$

Next derive case specific eq$\underline{^{ns}}$

Here make use of the linear eq$\underline{^{ns}}$

$$m(x)\phi'(x) = A(x)\phi(x) + B(x)\psi(x)$$

$$m(x)\psi'(x) = -C(x)\phi(x) - A(x)\psi(x)$$

$$\phi(x) = \sqrt{\xi}\,Ai(x), \quad \psi(x) = \sqrt{\xi}\,Ai'(x)$$

$$m(x) = 1, \quad A(x) = 0, \quad B(x) = 1, \quad C(x) = -x$$

One finds

$$\begin{aligned} q' &= p - qu \\ &= sq + pu - 2qv \\ &= pq' - qp' \\ &= -q^2 \end{aligned} \qquad \left\{ \begin{array}{l} a_1 = s \\ a_2 = \infty \\ p := p_1 \\ q := q_1 \\ R := R(s,s) \end{array} \right.$$

Supplement by the universal eq$\underline{^{ns}}$

$$u' = -q^2$$

$$v' = -pq$$

$$R = \frac{d}{ds} \log E((s,\infty);\xi)$$

From the final eq^{n} in each list

$$E((s,\infty)\ \ )\qquad \exp\left(\int_s^\infty R(t\ \ s)\,dt\right)$$

$$\exp\left(-\int_s^\infty \ t\ \ s)\,R(t,s)\,dt\right)$$

$$\exp\left(\int_t \ t-s)\,q^2(t\ \ s)\,dt\right)$$

A so by e iminat g var bles

a gst the ther eq^{s} find

$$(R)^2 + 4R'((R')^2\ \ sR + R) - 0$$

$$q'' = sq + 2q^3$$