Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

RMT moment calculations I

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Cambridge Lectures (1)

Gaussian Unitary Ensemble (GUE)

H – $N\times N$ hermitian matrix ($H_{nm} = H_{mn}^*$)

eigenvalues $E_n \in \mathbb{R}$.

Probability $P(H)\,dH = a\, e^{-b\,\mathrm{Tr}H^2}\, dH$.

Joint eigenvalue distribution function:

$$P_\beta(E_1, E_2, \ldots E_N)\prod_{n=1}^{N} dE_n$$

$$\propto e^{-b\sum_{n=1}^{N}E_n^2} \prod_{1\le j<k\le N} |E_j - E_k|^2\, dE_1 \ldots dE_N$$

Circular Unitary Ensemble (CUE)

A – $N\times N$ unitary matrix ($A(A^T)^* = I$) $= AA^\dagger$

ie $A \in U(N)$.

eigenvalues $e^{i\theta_n}$ $\theta_n \in \mathbb{R}$.

Probability – uniform density wrt Haar measure on $U(N)$.

Def[n] $f(A) = f(\theta_1, \theta_2, \ldots, \theta_N)$ is called a class function if f is symmetric in all of its variables

Weyl's integration formula (Weyl 1946)

for class functions

$$\int_{U(N)} f(A)\, d\mu_{Haar}(A) = \frac{1}{(2\pi)^N N!}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\theta_1,\dots,\theta_N)$$

$$\prod_{1\le j<k\le N} |e^{i\theta_j}-e^{i\theta_k}|^2\, d\theta_1\dots d\theta_N$$

[idea $d\mu_{Haar}(A)$ invariant under $A \to UAU^+$

also $A = U\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_N}\end{pmatrix}U^+$]

Two-Point Correlations

$$\phi_n = \theta_n \frac{N}{2\pi}$$

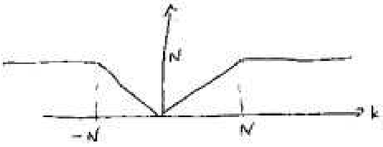
$$R_2(A;x) = \frac{1}{N}\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{k=-\infty}^{\infty}\delta(x+kN-\phi_n+\phi_m)$$

ie $$\frac{1}{N}\sum_{n,m} f(\phi_n-\phi_m) = \int_0^N R_2(A;x)\, f(x)\,dx.$$

$$R_2(A;x) = \frac{1}{N^2}\sum_{k=-\infty}^{\infty} |\mathrm{Tr}\, A^k|^2 e^{-2\pi i k x/N}$$

<u>Th^{m}</u> (Dyson 1963)

$$\int_{U(N)} |\operatorname{Tr} A^k|^2 \, d\mu_{Haar}(A) = \begin{cases} N^2 & k=0 \\ |k| & |k| \le N \\ N & |k| > N \end{cases}$$



<u>Lemma</u> for f a class function

$$\int_{U(N)} f(\theta_1, \ldots, \theta_N) \, d\mu(A) = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_N) \det\left(e^{i\theta_n (n-m)}\right) \, d\theta_1 \ldots d\theta_N$$

<u>Proof</u>

$$= \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1 \ldots \theta_N) \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \, d\theta_1 \ldots d\theta_N$$

and

$$\prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 = \det\left[\begin{pmatrix} 1 & 1 & \cdots \\ e^{i\theta_1} & e^{i\theta_2} & \cdots \\ e^{2i\theta_1} & e^{2i\theta_2} & \cdots \\ \vdots & & \\ e^{i(N-1)\theta_1} & \cdots & \end{pmatrix} \begin{pmatrix} 1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots \\ 1 & e^{-i\theta_2} & e^{-2i\theta_2} & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}\right]$$

$$= \det\left[\sum_{\ell=1}^{N} e^{i\theta_\ell (n-m)}\right]$$

$$\int_{U(N)} f(\theta_1, \theta_2, \dots, \theta_N)\, d\mu_{Haar}(A) =$$

$$\frac{1}{(2\pi)^N N!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \dots, \theta_N) \begin{vmatrix} \sum_{\ell=1}^N 1 & \sum_{\ell=1}^N e^{-i\theta_\ell} & \sum_{\ell=1}^N e^{-2i\theta_\ell} & \cdots \\ \sum_{\ell=1}^N e^{i\theta_\ell} & \sum_{\ell=1}^N 1 & \sum_{\ell=1}^N e^{-i\theta_\ell} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

$$= \quad \cdots \quad \cdots \quad N \begin{vmatrix} 1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots \\ \sum_{\ell=1}^N e^{i\theta_\ell} & \sum_{\ell=1}^N 1 & \sum_{\ell=1}^N e^{-i\theta_\ell} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

$$= \quad \cdots \quad \cdots \quad N \begin{vmatrix} 1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots \\ \sum_{\ell=2}^N e^{i\theta_\ell} & \sum_{\ell=2}^N 1 & \sum_{\ell=2}^N e^{-i\theta_\ell} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

QED.

$$\int_{U(N)} |\mathrm{Tr} A^k|^2 d\mu_{Haar}(A) = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_p \sum_q e^{ik(\theta_p - \theta_q)} \times$$

$$\times \det \begin{pmatrix} 1 & e^{-i\theta_1} & e^{-2i\theta_1} & \cdots & e^{-(N-1)i\theta_1} \\ e^{i\theta_2} & 1 & e^{-i\theta_2} & \cdots & e^{-(N-2)i\theta_2} \\ \vdots & & & & \\ e^{i(N-1)\theta_N} & e^{i(N-2)\theta_N} & & \cdots & 1 \end{pmatrix}$$

$$d\theta_1 \dots d\theta_N$$

$\sum$ diagonal ($p=q$) terms $=$ N

$k \geq N$ off-diagonal terms $= 0$. $\Rightarrow$ Thm for $k \geq N$

$k = N-j$ ~~off~~ " " $= -j$ $\Rightarrow$ Thm for $|k|<N$.

Hence

$$\int_{U(N)} R_2(A;x)\,d\mu(A) = \frac{1}{N^2}\sum_{k=-\infty}^{\infty} e^{-2\pi i kx/N} \begin{cases} N^2 & k=0 \\ |k| & |k|<N \\ N & |k|\geq N \end{cases}$$

$$= \sum_{j=-\infty}^{\infty} \delta(x-jN) + 1 - \frac{\sin^2 \pi x}{N^2 \sin^2\left(\frac{\pi x}{N}\right)}$$

so for $f(x) \to 0$ as $|x| \to \infty$

$$\lim_{N\to\infty} \int_{U(N)} \int_{-\infty}^{\infty} f(x)\, R_2(A;x)\,dx\,d\mu(A)$$

$$= \int_{-\infty}^{\infty} f(x)\left(\delta(x) + 1 - \frac{\sin^2 \pi x}{\pi^2 x^2}\right)dx$$

eg $\displaystyle \lim_{N\to\infty} \int_{U(N)} \frac{1}{N} \# \{ \phi_n, \phi_m : \alpha \leq \phi_n - \phi_m < \beta \}\, d\mu(A)$

$$= \int_{\alpha}^{\beta} \left(\delta(x) + 1 - \frac{\sin^2 \pi x}{\pi^2 x^2}\right) dx.$$

$\to$ Montgomery's conjecture.