Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Statistics of low-lying zeros of L-function and random matrix theory I

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DEFINITIONS

$$U = (u_{jk}) \qquad U^t = (u_{kj}) \qquad U^* = (\overline{u_{kj}})$$

$$\begin{aligned} U(N) &= \{U \text{ an } N \times N \text{ matrix } : UU^* = I\} \\ SO(N) &= \{R \in U(N) : RR^t = I, \det R = 1\} \\ USp(2N) &= \{P \in U(2N) : PZP^t = Z\} \end{aligned}$$

$$Z = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

EIGENVALUES

$$
\begin{array}{rll}
U(N): & e^{i\theta_1}, \ldots, e^{i\theta_N} & 0 \le \theta_j < 2\pi \\
SO(2N): & e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N} & 0 \le \theta_j \le \pi \\
SO(2N+1): & 1, e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N} & 0 \le \theta_j \le \pi \\
USp(2N): & e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N} & 0 \le \theta_j \le \pi
\end{array}
$$

We will use dU_N as a shorthand for the Haar measure on $U(N)$;
dP_N for the Haar measure on $USp(2N)$;
dR_N^+ for the Haar measure on $SO(2N)$ and dR_N^- for $SO(2N+1)$

So R is for "R-thogonal"

HAAR MEASURES

$$dU_N = \frac{1}{(2\pi)^N N!} \prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \dots d\theta_N$$

$$dR_N^+ = \frac{2^{(N-1)^2}}{\pi^N N!} \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \dots d\theta_N$$

$$dR_N^- = \frac{2^{N^2}}{\pi^N N!} \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \prod_{n=1}^{N} \sin^2\frac{\theta_n}{2} d\theta_n$$

$$dP_N = \frac{2^{N^2}}{\pi^N N!} \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 \prod_{n=1}^{N} \sin^2\theta_n d\theta_n$$

WEYL INTEGRATION FORMULAE

If $f(R)$ is a class function on $SO(2N)$ (i.e. it depends only on the set of eigenvalues of R), then

$$\int_{SO(2N)} f(R)\ dR_N^+$$

$$= \frac{2^{(N-1)^2}}{\pi^N N!} \int_{[0,\pi]^N} f(\theta_1, \ldots, \theta_N) \prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 d\theta_1 \ldots d\theta_N.$$

VANDERMONDE DETERMINANTS

For complex numbers $(x_1, \ldots, x_N)$ let

$$\Delta(x_1, \ldots, x_N) = \det_{N \times N} \left(x_k^{j-1} \right)$$

Then

$$\Delta(x_1, \ldots, x_N) = \prod_{1 \leq j < k \leq N} (x_k - x_j).$$

Proof: Both sides are homogeneous polynomials of total degree $N(N-1)/2$ which vanish whenever $x_j = x_k$. This fact identifies the two sides up to a constant factor. The coefficient of $x_N^{N-1} x_{N-1}^{N-2} \ldots x_2$ is 1 in both expressions.

VANDERMONDES CONT'D

Observe that

$$\prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^2 = |\Delta(e^{i\theta_1}, \dots, e^{i\theta_N})|^2$$

$$= \left(\det_{N\times N} (e^{i(j-1)\theta_k}) \right)^2$$

and

$$\prod_{1\le j<k\le N} (\cos\theta_k - \cos\theta_j)^2 = \Delta(\cos\theta_1, \dots, \cos\theta_N)^2$$

$$= \left(\det_{N\times N} (\cos^{j-1}\theta_k) \right)^2$$

ANDREIF'S IDENTITY

For any interval J and integrable functions ϕ_j and ψ_j:

$$\frac{1}{N!}\int_{J^N} \det_{N\times N}(\phi_j(\theta_k)) \det_{N\times N}(\psi_j(\theta_k))\ d\theta_1 \ldots d\theta_N$$

$$=$$

$$\det_{N\times N}\Big(\int_J \phi_j(\theta)\psi_k(\theta)\ d\theta\Big)$$

MASS OF THE HAAR MEASURE OF U(N)

Using

$$\phi_j(\theta) = e^{i(j-1)\theta}$$

we see that

$$\begin{aligned}\int_{[0,2\pi]^N} dU_N &= \int_{[0,2\pi]^N} |\det_{N\times N}(e^{i(j-1)\theta_k})|^2 \frac{d\theta_1 \ldots d\theta_N}{N!(2\pi)^N} \\ &= \frac{1}{(2\pi)^N} \det_{N\times N}\Big(\int_0^{2\pi} e^{i(j-1)\theta} e^{-i(k-1)\theta}\, d\theta\Big) \\ &= \frac{1}{(2\pi)^N} \det_{N\times N}\big(2\pi I\big) = 1\end{aligned}$$

This verifies that the total mass of the Haar measure of $U(N)$ is 1.

ORTHOGONAL POLYNOMIALS

$$\begin{aligned} T_n(\cos\theta) &:= \cos n\theta \\ U_n(\cos\theta) &:= \frac{\sin(n+1)\theta}{\sin\theta} \\ V_n(\cos\theta) &:= \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} \end{aligned}$$

$$T_n(x) = 2^{n-1}x^n + \dots \quad U_n(x) = 2^n x^n + \dots \quad V_n(x) = 2^n x^n + \dots$$

$$T_0^*(x) := 1 \qquad T_n^*(x) := \sqrt{2}T_n(x) \quad (n > 1)$$

T^*, U, V orthogonal on $[0,\pi]$ w.r.t. $d\theta$, $\sin^2\theta\ d\theta$, $\sin^2\frac{\theta}{2}\ d\theta$

Each vector has the same norm: $||T_j^*||^2 = \pi, ||U_j||^2 = \frac{\pi}{2}, ||V_j||^2 = \frac{\pi}{2}$.

REWRITING THE VANDERMONDE

By elementary row operations

$$\begin{aligned}\Delta(\cos\theta_1,\ldots,\cos\theta_N)&\\ &= 2^{-(N-1)^2/2}\det_{N\times N}(T^*_{j-1}(\cos\theta_k))\\ &= 2^{-N(N-1)/2}\det_{N\times N}(U_{j-1}(\cos\theta_k))\\ &= 2^{-N(N-1)/2}\det_{N\times N}(V_{j-1}(\cos\theta_k))\end{aligned}$$

HAAR MEASURES AS SQUARES OF DETS

$$
\begin{aligned}
dU_N &= \frac{1}{(2\pi)^N N!}\left| \det_{N\times N}(e^{i(j-1)\theta_k})\right|^2 d\theta_1 \dots d\theta_N \\
dR_N^+ &= \frac{1}{\pi^N N!}\left(\det_{N\times N}(T^*_{j-1}(\cos\theta_k)\right)^2 d\theta_1 \dots d\theta_N \\
dR_N^- &= \frac{2^N}{\pi^N N!}\left(\det_{N\times N}(V_{j-1}(\cos\theta_k)\right)^2 \prod_{n=1}^{N} \sin^2\frac{\theta_n}{2}\; d\theta_n \\
dP_N &= \frac{2^N}{\pi^N N!}\left(\det_{N\times N}(U_{j-1}(\cos\theta_k)\right)^2 \prod_{n=1}^{N} \sin^2\theta_n\; d\theta_n
\end{aligned}
$$

MASS OF dR_N^+

$$\int_{[0,\pi]^N} dR_N^+ = \frac{1}{\pi^N N!} \det_{N\times N}\left(\int_0^\pi T_{j-1}^*(\cos\theta) T_{k-1}^*(\cos\theta)\, d\theta \right) = 1,$$

since

$$\int_0^\pi T_{j-1}^*(\cos\theta) T_{k-1}^*(\cos\theta)\, d\theta = \pi \delta_{j,k}$$

GENERALIZED ANDREIF

$$\frac{1}{N!}\int_{J^N}\prod_{i=1}^{N} f(\theta_i)\,\det_{N\times N}(\phi_j(\theta_k))\,\det_{N\times N}(\psi_j(\theta_k))\; d\theta_1\ldots d\theta_N$$
$$= \det_{N\times N}\Big(\int_J f(\theta)\phi_j(\theta)\psi_k(\theta)\; d\theta\Big)$$

MASS OF dR_N^- AND dP_N

$$\int_{SO(2N+1)} dR_N^- = \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi V_{j-1}(\cos\theta)\, V_{k-1}(\cos\theta)\sin^2\frac{\theta}{2}\, d\theta\right) = 1$$

and

$$\int_{USp(2N)} dP_N = \frac{2^N}{\pi^N} \det_{N\times N}\left(\int_0^\pi U_{j-1}(\cos\theta)\, U_{k-1}(\cos\theta)\sin^2\theta\, d\theta\right) = 1$$

since the integrals are $\frac{\pi}{2}$ when $j = k$ and 0 otherwise.

PROOF OF ANDREIF'S IDENTITY

$$\int_{J^N} \det_{N\times N}\big(\phi_j(\theta_k)\big) \det_{N\times N}\big(\psi_j(\theta_k)\big) d\theta_1 \dots\ d\theta_N$$

$$= \quad \int_{J^N} \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \sum_{\tau} \mathrm{sgn}(\tau) \prod_{k=1}^{N} \psi_k(\theta_{\tau k}) \ \prod_{i=1}^{N} d\theta_i$$

$$\underset{\tau\to\sigma\tau}{=} \quad \int_{J^N} \sum_{\sigma,\tau} \mathrm{sgn}(\tau) \prod_{j,k=1}^{N} \phi_j(\theta_{\sigma j})\psi_k(\theta_{\sigma\tau k}) \ \prod_{i=1}^{N} d\theta_i$$

$$\underset{k\to\tau^{-1}k}{=} \quad \int_{J^N} \sum_{\sigma,\tau} \mathrm{sgn}(\tau) \prod_{j,k=1}^{N} \phi_j(\theta_{\sigma j})\psi_{\tau^{-1}k}(\theta_{\sigma k}) \ \prod_{i=1}^{N} d\theta_i$$

$$
\begin{aligned}
&= \quad \int_{J^N} \sum_{\sigma,\tau} \mathrm{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau^{-1} j}(\theta_{\sigma j}) \ \prod_{i=1}^{N} d\theta_i \\
&\underset{\tau \to \tau^{-1}}{=} \quad \int_{J^N} \sum_{\sigma,\tau} \mathrm{sgn}(\tau) \prod_{j=1}^{N} \phi_j(\theta_{\sigma j}) \psi_{\tau j}(\theta_{\sigma j}) \ \prod_{i=1}^{N} d\theta_i \\
&= \quad \sum_{\sigma,\tau} \mathrm{sgn}(\tau) \prod_{j=1}^{N} \int_J \phi_j(\theta) \psi_{\tau j}(\theta) \ d\theta \\
&= \quad N! \sum_{\tau} \mathrm{sgn}(\tau) \prod_{j=1}^{N} \int_J \phi_j(\theta) \psi_{\tau j}(\theta) \ d\theta \\
&= \quad N! \det_{N \times N} \big(\int_J \phi_j(\theta) \psi_k(\theta) \ d\theta \big).
\end{aligned}
$$

TRANSPOSING LEMMA

$$\det_{N\times N}(\phi_{j-1}(x_k))\,\det_{N\times N}(\psi_{j-1}(x_k)) = \det_{N\times N}\big(\sum_{n=1}^{N}\phi_{n-1}(x_j)\psi_{n-1}(x_k)\big).$$

This identity just follows by using the fact that the determinant of a matrix and its transpose are the same, and matrix multiplication:

$$\begin{aligned}&\det_{N\times N}(\phi_{j-1}(x_k))\,\det_{N\times N}(\psi_{j-1}(x_k))\\ &\quad= \det_{N\times N}(\phi_{j-1}(x_n))_{j,n}\,\det_{N\times N}(\psi_{n-1}(x_k))_{n,k}\\ &\quad= \det\big(\sum_{n=1}^{N}\phi_{n-1}(x_j)\psi_{n-1}(x_k)\big)_{j,k}\end{aligned}$$

TRANSPOSING THE SQUARE DETS IN THE MEASURES

$$\left|\det_{N\times N} e^{i(j-1)\theta_k}\right|^2 = \det_{N\times N}\left(\sum_{n=1}^{N} e^{i(n-1)\theta_j} e^{-i(n-1)\theta_k}\right)$$

$$\left(\det_{N\times N}\left(T^*_{j-1}(\cos\theta_k)\right)\right)^2 = \det_{N\times N}\left(\sum_{n=1}^{N} T^*_{n-1}(\cos\theta_j) T^*_{n-1}(\cos\theta_k)\right)$$

$$= \det_{N\times N}\left(1 + 2\sum_{n=1}^{N-1} \cos n\theta_j \cos n\theta_k\right)$$

$$\left(\det_{N\times N}\left(U_{j-1}(\cos\theta_k)\right)\right)^2 \prod_{n=1}^{N} \sin^2\theta_n = \det_{N\times N}\left(\sum_{n=1}^{N} \sin n\theta_j \sin n\theta_k\right)$$

$$\left(\det_{N\times N}\left(V_{j-1}(\cos\theta_k)\right)\right)^2 \prod_{n=1}^{N} \sin^2\frac{\theta_n}{2} = \det_{N\times N}\left(\sum_{n=1}^{N} \sin(n-\tfrac{1}{2})\theta_j \sin(n-\tfrac{1}{2})\theta_k\right)$$

HAAR MEASURES AS DETS

$$
\begin{aligned}
dU_N &= \frac{1}{(2\pi)^N N!} \det_{N\times N} \left(\sum_{n=1}^{N} e^{i(n-1)\theta_j} e^{-i(n-1)\theta_k} \right) d\theta_1 \ldots d\theta_N \\
dR_N^+ &= \frac{1}{\pi^N N!} \det_{N\times N} \big(1 + 2 \sum_{n=1}^{N-1} \cos n\theta_j \cos n\theta_k \big) d\theta_1 \ldots d\theta_N \\
dR_N^- &= \frac{2^N}{\pi^N N!} \det_{N\times N} \big(\sum_{n=1}^{N} \sin(n-\tfrac{1}{2})\theta_j \sin(n-\tfrac{1}{2})\theta_k \big) d\theta_1 \ldots d\theta_N \\
dP_N &= \frac{2^N}{\pi^N N!} \det_{N\times N} \big(\sum_{n=1}^{N} \sin n\theta_j \sin n\theta_k \big) d\theta_1 \ldots d\theta_N
\end{aligned}
$$

TRIGONOMETRIC SUMS

$$S_N(\theta) := \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\sum_{n=1}^{N} e^{i(n-1)\theta_j} e^{-i(n-1)\theta_k} = e^{-iN(\theta_k-\theta_j)/2} S_N(\theta_k - \theta_j)$$

$$1 + 2\sum_{n=1}^{N-1} \cos nx \cos ny = (S_{2N-1}(y-x) + S_{2N-1}(y+x))/2$$

$$2\sum_{n=1}^{N} \sin nx \sin ny = (S_{2N+1}(y-x) - S_{2N+1}(y+x))/2$$

$$2\sum_{n=1}^{N} \sin(n-\tfrac{1}{2})x \sin(n-\tfrac{1}{2})y = (S_{2N}(y-x) - S_{2N}(y+x))/2$$

ALTERNATE FORMULAS FOR HAAR MEASURES

$$
\begin{aligned}
S_{U,N}(x,y) &:= S_N(y-x)\\
S_{R^+,N}(x,y) &:= \frac{1}{2}(S_{2N-1}(y-x)+S_{2N-1}(y+x))\\
S_{R^-,N}(x,y) &:= \frac{1}{2}(S_{2N}(y-x)-S_{2N}(y+x))\\
S_{P,N}(x,y) &:= \frac{1}{2}(S_{2N+1}(y-x)-S_{2N+1}(y+x))
\end{aligned}
$$

ALTERNATE FORMULAS CONT'D

$$
\begin{aligned}
dU_N &= \det_{N\times N}(S_{U,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{(2\pi)^N N!} \\
dR_N^+ &= \det_{N\times N}(S_{R^+,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{\pi^N N!} \\
dR_N^- &= \det_{N\times N}(S_{R^-,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{\pi^N N!} \\
dP_N &= \det_{N\times N}(S_{P,N}(\theta_j,\theta_k))\frac{d\theta_1\ldots d\theta_N}{\pi^N N!}
\end{aligned}
$$

GAUDIN'S LEMMA

Suppose that we have a function f and a measurable set J such that

$$\int_J f(x,\theta) f(\theta,y)\ d\theta = C f(x,y)$$

for all x and y where $C = C(J,f)$ is a constant, and

$$\int_J f(x,x)\ dx = D.$$

Then,

$$\int_J \det_{M\times M} \left(f(\theta_j,\theta_k)\right)\ d\theta_M = (D-(M-1)C) \det_{M-1} \left(f(\theta_j,\theta_k)\right)$$

GAUDIN'S LEMMA AND $S_{G,N}$

Let G denote R^+, R^-, or P. Then

$$\int_{[0,\pi]} S_{G,N}(x,\theta)S_{G,N}(\theta,y)\ d\theta = \pi S_{G,N}(x,y)$$

$$\int_0^\pi S_{G,N}(x,x)\ dx = \pi N$$

Also

$$\int_{[0,2\pi]} S_{U,N}(x,\theta)S_{U,N}(\theta,y)\ d\theta = 2\pi S_{U,N}(x,y)$$

$$\int_0^{2\pi} S_{U,N}(x,x)\ dx = 2\pi N$$

GAUDIN'S LEMMA AND $S_{G,N}$

For $G = R^+, R^-, P$:

$$\int_{[0,\pi]} \det_{M\times M} S_{G,N}(\theta_j,\theta_k)\ d\theta_M = \pi(N-(M-1)) \det_{M-1} S_{G,N}(\theta_j,\theta_k)$$

Applying this with $M = N$, then $M = N-1$, …, then $M = n+1$:

$$\int_{[0,\pi]^{N-n}} \det_{N\times N} S_{G,N}(\theta_j,\theta_k)\ d\theta_{n+1}\dots d\theta_N$$
$$= \pi^{N-n}(N-n)! \det_{n\times n} S_{G,N}(\theta_j,\theta_k)$$

Similarly,

$$\int_{[0,2\pi]^{N-n}} \det_{N\times N} S_{U,N}(\theta_j,\theta_k)\ d\theta_{n+1}\dots d\theta_N$$
$$= (2\pi)^{N-n}(N-n)! \det_{n\times n} S_{U,N}(\theta_j,\theta_k)$$

PROOF OF REPRODUCING PROPERTY FOR $S_{P,N}$

$$S_{P,N}(x,y) = 2\sum_{n=1}^{N} U_{n-1}(\cos x)U_{n-1}(\cos y)\sin x \sin y$$

$$\int_0^\pi S_{P,N}(x,\theta)S_{P,N}(\theta,y)\ d\theta = 4\sin x \sin y$$

$$\times \int_0^\pi \sum_{m,n=0}^{N-1} U_n(\cos x)U_n(\cos\theta)U_m(\cos\theta)U_m(\cos y)\sin^2\theta\ d\theta$$

$$= 4\sin x \sin y \frac{\pi}{2}\sum_{n=0}^{N-1} U_n(\cos x)U_n(\cos y)$$

$$= 2\pi\sum_{n=1}^{N} U_{n-1}(\cos x)U_{n-1}(\cos y)\sin x \sin y = \pi S_{P,N}(x,y)$$

USING GAUDIN'S LEMMA TO COMPUTE n-LEVEL DENSITY

Let $B \subset \{1, \ldots, N\}$, $|B| = n$. Let f be a symmetric function of n vbles. Let $f(\theta_B) = f(\theta_{b_1}, \ldots, \theta_{b_n})$ where $B = \{b_1, \ldots, b_n\}$. Then

$$\int_{U(N)} \sum_{\substack{B \subset [1,N] \\ |B| = n}} f(\theta_B) dU_N$$
$$= \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} f(\theta_1, \ldots, \theta_n) \det_{n \times n}(S_{U,N}(\theta_j, \theta_k)) d\theta_1 \ldots d\theta_n$$

We call $\det_{n \times n}(S_{U,N}(\theta_j, \theta_k))$ the n-level density function for $U(N)$.

THE OTHER n-LEVEL DENSITIES

Let $G = R^+, R^-$, or P. Then

$$\int_{G,N} \sum_{\substack{B\subset[1,N]\\|B|=n}} f(\theta_B)dG_N$$

$$= \frac{1}{\pi^n n!}\int_{[0,\pi]^n} f(\theta_1,\ldots,\theta_n) \det_{n\times n}(S_{G,N}(\theta_j,\theta_k))d\theta_1\ldots d\theta_n$$

We call $\det_{n\times n}(S_{G,N}(\theta_j,\theta_k))$ the n-level density function for G, N.

LARGE N LIMIT OF THE n-LEVEL DENSITY

Suppose that we have an infinite sequence $X : 0 \leq x_1, x_2, \ldots$ whose average spacing is 1, i.e. $\lim_{N\to\infty}(x_N - x_1)/N = 1$. Suppose that f is a symmetric function of n variables which is rapidly decaying as one leaves the origin. For each subset
$B = \{b_1, \ldots, b_n\} \subset \{1, \ldots, N\}$ of size n we evalate f at the n-tuple of points whose indices are in B. We add all of these up and take the limit as $N \to \infty$. If this limit exists and if, for some K,

$$\lim_{N\to\infty} \sum_{\substack{B\subset[1,N]\\|B|=n}} f(x_B) = \frac{1}{n!}\int_{R_+^n} f(x_1, \ldots, x_n)K(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

for all reasonable f, then we call $K(x_1, \ldots, x_n)$ the n-level density function associated with X.

NORMALIZED EIGENANGLES FOR $U(N)$

For $U \in U(N)$ with eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_N}$ let $\tilde{U} = \{\frac{N\theta_1}{2\pi}, \dots, \frac{N\theta_N}{2\pi}\}$. These N numbers are all contained in $[0, N]$. Letting $\tilde{\theta} = N\theta/(2\pi)$ we calculate

$$\lim_{N\to\infty} \int_{U(N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f(\tilde{\theta}_B) dU_N$$

$$= \lim_{N\to\infty} \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} f(\tilde{\theta}_1, \dots, \tilde{\theta}_n) \det_{n\times n}(S_{U,N}(\theta_j, \theta_k)) d\theta_1 \dots d\theta_n$$

$$= \lim_{N\to\infty} \frac{1}{n!} \int_{[0,N]^n} f(\theta_1, \dots, \theta_n) \det_{n\times n} \left(\frac{S_N(2\pi(\theta_k - \theta_j))/N}{N} \right) \prod_{j=1}^n d\theta_j$$

$$= \frac{1}{n!} \int_{R_+^n} f(\theta_1, \dots, \theta_n) \det_{n\times n} \left(S(\theta_k - \theta_j) \right) \prod_{j=1}^n d\theta_j$$

since $\lim_{N\to\infty} S_N(2\pi x/N)/N = \lim \sin(\pi x)/(N \sin \pi x/N) = S(x)$

NORMALIZED EIGENANGLES FOR $USp(N)$

For $P \in USp(2N)$ with eigenvalues $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N}$ and $0 \le \theta_j \le \pi$, let $\tilde{P} = \{\frac{N\theta_1}{\pi}, \dots, \frac{N\theta_N}{\pi}\}$. These N numbers are all contained in $[0, N]$. Letting $\tilde{\theta} = N\theta/\pi$ we calculate

$$\lim_{N\to\infty} \int_{USp(2N)} \sum_{\substack{B\subset[1,N]\\|B|=n}} f(\tilde{\theta}_B) dP_N$$

$$= \lim_{N\to\infty} \frac{1}{\pi^n n!} \int_{[0,\pi]^n} f(\tilde{\theta}_1, \dots, \tilde{\theta}_n) \det_{n\times n}(S_{P,N}(\theta_j, \theta_k)) d\theta_1 \dots d\theta_n$$

$$= \lim_{N\to\infty} \frac{1}{n!} \int_{[0,N]^n} f(\theta_1, \dots, \theta_n) \det_{n\times n} \left(\frac{S_{P,N}(\pi\theta_j/N, \pi\theta_k/N)}{N} \right) \prod_{j=1}^{n} d\theta_j$$

$$= \frac{1}{n!} \int_{R_+^n} f(\theta_1, \dots, \theta_n) \det_{n\times n} \left(S(\theta_k - \theta_j) - S(\theta_j + \theta_k) \right) \prod_{j=1}^{n} d\theta_j$$

since $\lim_{N\to\infty} S_{2N+1}(\pi x/N)/(2N) =$
$\lim \sin(\pi x(1 + 1/(2N)))/(2N \sin \pi x/(2N)) = S(x)$

DENSITY FUNCTIONS

$$\begin{aligned} n\text{-level density for U}: &= \det_{n\times n} S_U(\theta_j,\theta_k)\text{where} \\ S_U(x,y) &= S(y-x) \\ n\text{-level density for G}: &= \det_{n\times n} S_G(\theta_j,\theta_k)\text{where} \\ S_{R^+}(x,y) &= S(y-x)+S(y+x) \\ S_P(x,y) &= S(y-x)-S(y+x) \end{aligned}$$

n-level density for R^-:

$$= \det_{n\times n} S_P(\theta_j,\theta_k) + \sum_{m=1}^{n} \delta(\theta_m) \det_{n-1} S_P^{(m)}(\theta_j,\theta_k)$$

where $S_P^{(m)}$ means the mth row and mth column omitted