Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Low moments of the Riemann zeta-function

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Low Moments of the Riemann zeta function

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Computing moments of $\zeta(\frac{1}{2}+ t)$

$$\zeta(\qquad \sum_{n \quad T} \qquad \frac{T^{1-s}}{1-s} + O(T^{\sigma}) \qquad t/T$$

$$\zeta(\tfrac{1}{2}+it = \sum_{n \quad T} \frac{}{n^{\frac{1}{2}+ t}} + O\left(\frac{T^{\frac{1}{2}}}{+t}\right)$$

$$\int_0^T \zeta(\tfrac{1}{2}+it)\,dt = \sum_{n \quad T} \frac{}{\sqrt{n}} \int_0^T n \quad dt + O\left(T^{\frac{1}{2}} \int_0^T \frac{dt}{+it}\right)$$

$$T + \sum_{2 \le n \quad T} \frac{}{\sqrt{n}} \frac{n^{\quad T}}{\log n} + O(T^{1/2}\log T$$

$$= T + O(T^{\frac{1}{2}} \log T)$$

Not best p ssible error term

Not r a y a "moment of $\zeta(\frac{1}{2}+ t$

Moments of Dirichlet polynomials

$$\int_0^T \left| \sum_{\leq n \, N} \frac{a_n}{n^{it}} \right|^2 dt \quad \int_0^T \sum_n \frac{a_n}{} + \sum_m \frac{a}{m} \quad dt$$

$$- \sum_{m \, n} a_n a_m \int_0^T \left(\frac{m}{} \right)^t dt$$

$$- T \sum_n a_n^2 + \sum_{m \neq n} \frac{a_n a_m \left(\left(\frac{m}{n} \right)^T \right)}{\log (m/n}$$

$$M(T \qquad + E(T)$$

$$E(T) = \sum_n \sum_{h \neq 0} \frac{a_n a_{+h} \left(\left(+ \frac{h}{} \right)^T \right)}{\log \left(+ \frac{h}{n} \right)}$$

h - terms of size

$$\sum_{\leq n \, N} a_n a_{n+} \quad n$$

Could be arger than $M(T)$ f $N > T$

Mean value therem of Montgomery and Vaughn

$$\int^{T} \sum_{n} \frac{a_n}{n^{t}} \Big|^{2} dt \quad \sum_{n} a_n \left(T + O(n \right)$$

$$\zeta(\tfrac{1}{2} + t \quad \sum_{\le n\, T} \frac{1}{n^{1/2}\, t} + \quad O\left(\frac{T^{\frac{1}{2}}}{t|t} \right)$$

$$= \quad S \quad + \quad E$$

$$\int_0^T \zeta(\tfrac{1}{2} + t \quad ^{2} dt = \int^{T} s^{2} + \underbrace{2\,\mathrm{Re} \int_0^T S E}_{\ll \int_0^T s \ E\, dt} + \int_0^T E^{2} d$$

$$< \left(\int_0^T s^{2}\, dt \right)^{1/2} \left(\int_0^T |E|^{2} dt \right)^{1/2}$$

ok if $\int_0^T |E^{2}$

ok

$$\int_0^T \zeta(\tfrac{1}{2}+it)^2\,dt \quad \int^T S^2\,dt + \int^T E^2\,dt + \text{OK term}$$

$$\int_0^T |E|^2\,dt - T\int_0^T \frac{1}{(+|t|)}\,dt$$

$$\ll T$$

$$\int_0^T \Big|\sum_{T} \frac{}{n^{1/2+t}}\Big|^2 dt \quad \overset{m-v}{} \quad \sum_{\leq n\, T} \frac{}{n}\, T + O($$

$$T\log T + O\Big(\sum_{\leq n\leq T}\Big)$$

$$T\log T + O(T)$$

so

$$\int_0^T \zeta(\tfrac{1}{2}+it)^2\,dt \quad T\log T + O(T\log^{1/2} T)$$

Approximate functiona equation

$$\zeta(\;) - \sum_{n\;x} \frac{1}{n^s} + \chi(s) \sum_{n\;y} n^{1-s} + O\left(x^{\sigma} + |t|^{\frac{1}{2}-\sigma} y^{\sigma-1}\right)$$

Here $xy \;\; \frac{t}{2\pi}$

Recall $\zeta(s \;\; \chi(s\; \zeta(\;\; s)$

$$\zeta(\tfrac{1}{2} + t) \quad \sum_{\leq N} n^{1/2 \; it} + \chi(\tfrac{1}{2} + t) \sum_{n\;N} n^{\frac{1}{2}} \; t + O(N^{\frac{1}{2}})$$

$$S \quad + \chi(\tfrac{1}{2} + t)\; S \quad + \; E$$

S has length $\tfrac{?}{2}\, t^{1/2} \ll T^{1/2}$

$$\int^{T} \zeta(\tfrac{1}{2} + t)\;^{2} dt \quad 2 \int_0^T |S|^2 dt$$

$$+ 2\,\mathrm{Re} \int_0^T \chi(\tfrac{1}{2} \;\; t)\, S^2 dt$$

$+$ error

$\chi(\frac{1}{2}+it$

$$\chi(\frac{1}{2}+t - \left(\frac{2\pi e}{t}\right)^{t}\left(1+O(\frac{1}{t})\right)$$

Cross terms are

$$\sum_{n,m} \frac{}{\sqrt{nm}} \int_0^T \chi(\frac{1}{2} \quad t)(nm)^{\quad t}\, dt$$

$$\int^T \chi(\frac{1}{2}+t)\, m^{+it}\, dt \quad \int^T e^{if(t}\, dt$$

$$f(t) \quad t(\pm \log m \quad \log \frac{t}{2\pi})$$

$$f'(t) = \log \frac{t}{2\pi m} \text{ or } \log \frac{tm}{2\pi}$$

$$0 \text{ when } t = 2\pi m \text{ or } \frac{2\pi}{m}$$

So we can calculate the contribution of these terms

$$\int_0^T \zeta(\frac{1}{2}+t)^2\, dt \quad T\log T + CT + O(T^{\frac{1}{2}+\varepsilon})$$

c d
etter

(n

If $M(s) \sum_{n<T^\theta} \frac{b_n}{n^s}$ then

$\zeta(s\ M(s)$ can be written in terms of Dirichlet polynomials of length $T^{\frac{1}{2}+\theta}$

So by M V, can evaluate $\int_0^T |\zeta\, M(\tfrac{1}{2}+it)|^2$

easily f $\theta < {}_2$

Good reasons to expect $\theta <$ to work in reasonable generality)

(May have larger θ for speci l $M(s$)

$$\int^{T} \zeta(+ t) ^{4} dt \qquad {}_{2\pi} \; T \log^{4} T + \text{l.o.t} + \text{error}$$

approximate functional equation for $\zeta^{2}(s$

$$\zeta^{2}(- \sum_{n \; x} \frac{d}{n^{s}} + \chi^{2}(\sum_{n \leq} \frac{d \; n}{n^{\; s}} + O(\chi^{\frac{1}{2}-\sigma} \log t)$$

Here $xy = \left(\frac{t}{2\pi}\right)^{2}$

$d \; n$ # of divisors of n Cut $n = n \, n$

$$\zeta^{2}(s \qquad \sum_{n=}^{\infty} \frac{d(n}{n^{s}} \quad \text{for } \sigma >$$

$$\zeta^{k}(s - \sum_{n \; x} \frac{d_{k}(n}{n^{s}} + \chi^{k} \sum_{n \; y} \frac{d_{k} \, n}{n} + \text{error}$$

$d_{k} \, n \quad \sum_{n \;\; n_{k} = n}$ k fold divisor function

x and y of size $t^{k/2}$

+ $k > 2$

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Conjecture (Conrey and Ghosh)

$$\int_0^T \zeta(\tfrac{1}{2}+it)^{2k}\, dt \qquad \frac{g_k a_k}{k^2}\, T \log^{k^2} T$$

where g_k IS AN INTEGER

and $a_k \quad \prod_p \quad -p)^{k^2} \sum_{m=0}^{\infty} \binom{k+m}{m}^2 p^{\quad m}$

k	g_k	Theorem?	
0	1	Y	
1	1	Y	Hardy & Littlewood
2	2	Y	Ingham
3	42		Conrey-Ghosh, Conrey Gonek
4	24024		Conrey-Gonek

$$k^2 \prod_{j=0}^{k\ 1} \frac{j}{(k+\ )}$$ keating Snaith conjecture

By M V and the approximate functional eqn, you expect $\int_0^T |\zeta(\tfrac{1}{2}+t)|^{2k} dt$ to involve

$$\sum_{n<T} \frac{d_k n^2}{n} \qquad 2\pi i \int_{(C)} \sum_n^{\infty} \frac{d_k{}^2}{n^{s+}} \frac{T^s}{s} ds$$

Rec

$$\frac{1}{2\pi i}\int_{C} \frac{y^s}{s} ds \begin{cases} & y > \\ 0 & y < \end{cases}$$

$c >> 0$

use $y - \frac{T}{n} >$ f $n < T$

Since $d_k n << n^c$ expect that

$$L\ s \quad \sum_{n-}^{\infty} \frac{d_k(\ ^2}{n^s} \qquad \frac{d_k}{(S}\ N^{+}$$

$$\sum_{n<T} \frac{d_k(n^2}{n} - 2\pi \int_{(C)} L\ s+ \frac{T^s}{s} ds$$

$$a_k \log^N T$$

N

(7)

(11)

$$\zeta(s = \prod_p \left(1 + \frac{1}{p^s} + \qquad\right) \qquad s\ 1^+$$

$$\zeta(s)^N \quad \prod_p \left(\ + \frac{N}{p^s} + \qquad \right) \sim \frac{\quad}{(s\ 1)^N}$$

coeff c ent of $\ \ p^s$ tells you the order of the eading po e at $s =$

$$L(s \quad \prod_p \left(1 + \frac{d_k(p)^2}{p^s} + \frac{d_k(p\ )}{p^{2s}} + \qquad \right)$$

$$d_k(p) \quad \sum_{n \quad n_k = p} \qquad k \qquad \text{so } N \quad k^2$$

$$\text{so } L(\ \sim \frac{a_k}{(s\ \ )^{k^2}} +$$

$$a_k \quad \lim_{s \to} \frac{\quad}{\zeta(s)^{k^2}} L(s)$$

$$= \prod_p \left(\qquad p \right)^{k^2} \sum_{m=0}^{\infty} d_k\ p^m)^2\ p \ \ ^m$$

$$d\ (p^3) \quad {}^5_{\ 5}\ ) \qquad 1 \qquad p \qquad 1. \quad 1$$

<u>Theorem</u> (Goldston Gonek, and Montgomery)

If $a \sim 1/\log T$ and [a believe the hypothesis is true]

Then

$$\int_0^T \left| \frac{\zeta'}{\zeta}\left(\tfrac{1}{2}+a+it\right)\right|^2 dt$$

$$\sim T\log^2 T \int_0^\infty F(\alpha,T)\, T^{-2a\alpha}$$

$$\int^\infty \left(\psi\left(x+\frac{x}{T}\right) \quad \psi(x) - \frac{x}{T}\right)^2 x^2 \frac{dx}{x^2}$$

$$F(\alpha,T) \quad \frac{1}{N(T)} \sum_{0<\gamma'<T} T^{i\alpha(\gamma\ \gamma')} \quad w(\gamma\ \gamma$$

2 point for factor

$$\psi(x \quad \sum_{n<x} \Lambda(n \quad \sim x$$ prime counting function