Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

RMT moment calculations II

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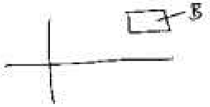
Cambridge Lectures (2) (JPK + NCS)

look at distributional properties of $\zeta(\frac{1}{2}+it)$, $\log \zeta(\frac{1}{2}+it)$

$\log \zeta(\frac{1}{2}+it)$

Th^m (Selberg)



$$\lim_{T\to\infty} \frac{1}{T}\,\text{meas.}\left\{ T \le t \le 2T : \frac{\log \zeta(\frac{1}{2}+it)}{\sqrt{\frac{1}{2}\log\log \frac{t}{2\pi}}} \in B \right\}$$

$$= \iint_B e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy.$$

$\boxed{\text{Odlyzko's numerics}}$

$\zeta(\frac{1}{2}+it)$

conjecture:

$$\lim_{T\to\infty} \frac{1}{(\log \frac{T}{2\pi})^{\lambda^2}} \frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2\lambda} dt = f_\zeta(\lambda) \prod_p \left[(1-\tfrac{1}{p})^{\lambda^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+\lambda)}{m!\,\Gamma(\lambda)}\right)^2 p^{-m}\right]$$

$f_\zeta(1)=1$ (HL 1918), $f_\zeta(2)=\frac{1}{12}$ (Ingham 1926), $f_\zeta(3) \overset{?}{=} \frac{42}{9!}$ C-G 1992,
$f_\zeta(4) = \frac{24024}{16!}$ C-G 1998.

Characteristic polynomials of random unitary matrices

– $Z(A,\theta) = \det(I - Ae^{-i\theta})$

$$= \prod_n \left(1 - e^{i(\theta_n - \theta)}\right)$$

moment generating function for $\log Z$:

$$P_N(s,t) = \int_{U(N)} |Z(A,\theta)|^t \, e^{is \operatorname{Im} \log Z(A,\theta)} \, d\mu(A)$$

- joint moments of $\operatorname{Re} \log Z$ + $\operatorname{Im} \log Z$ obtained from derivatives of P at $s=0, t=0$
- $\int_{U(N)} \delta(x - \operatorname{Re}\log Z)\, \delta(y - \operatorname{Im}\log Z)\, d\mu(A)$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itx - isy} \, P(s, it) \, ds\, dt$$

$$P_N(s,t) = \int_{U(N)} \underbrace{\prod_{n=1}^{N} |1 - e^{i(\theta_n - \theta)}|^t \, e^{-is \sum_{m=1}^{\infty} \frac{\sin[(\theta_n - \theta)m]}{m}}}_{\text{class } f^n} \, d\mu(A)$$

Weyl integration formula: $\int_0^{2\pi} \cdots \int_0^{2\pi} \cdots \, d\theta_1 \cdots d\theta_N$

Selberg integral (eg Mehta)

$$\Rightarrow \boxed{P_N(s,t) = \prod_{j=1}^{N} \frac{\Gamma(j)\,\Gamma(t+j)}{\Gamma(j+\frac{t}{2}+\frac{s}{2})\,\Gamma(j+\frac{t}{2}-\frac{s}{2})}}$$

③

Taylor expansion:

$$P_N(s,t) = e^{\alpha_{00} + \alpha_{10} t + \alpha_{01} s + \alpha_{20} t^2/2 + \alpha_{11} ts + \alpha_{02} s^2/2 + \cdots}$$

$$\cancel{\alpha_{mn} = \begin{cases} 0 & n\text{ - odd} \\ -\frac{1}{2^{n+m-1}} \sum_{j=1}^{N} \psi^{(n+m-1)}(j) & n\text{ - even} \end{cases}}$$

α_{m0} - cumulants of $\mathrm{Re}\log Z$

α_{0n} - $i^n \times ($ " " $\mathrm{Im}\log Z)$

$$\alpha_{10} = \alpha_{01} = \alpha_{11} = 0$$

$$\alpha_{20} = -\alpha_{02} = \frac{1}{2}\log N + \frac{1}{2}(\gamma+1) + O\left(\frac{1}{N^2}\right)$$

$$\alpha_{mn} = O(1) \quad \text{for} \quad m+n \geq 3$$

$$\left(\alpha_{m0} = (-1)^m \left(1 - \frac{1}{2^{m-1}}\right) \Gamma(m)\,\zeta(m-1) + O\left(\frac{1}{N^{m-2}}\right) \quad m \geq 3 \right)$$

$\therefore$ Th^{m} (K, Snaith)

$$\lim_{N\to\infty} \text{meas.} \left\{ A \in U(N) : \frac{\log Z(A,\theta)}{\sqrt{\frac{1}{2}\log N}} \in B \right\}$$

$$= \frac{1}{2\pi} \iint_B e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy$$

$\boxed{N = \log \frac{t}{2\pi} \quad ??}$ ④

Moments of $|Z|$

$$\int_{U(N)} |Z(\overset{*}{A}, 0)|^{2\lambda} \, d\mu(A) = P(0, 2\lambda)$$

$$= \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\lambda)}{(\Gamma(j+\lambda))^2}$$

$$= e^{\sum_{n=0}^{\infty} \alpha_{n0} (2\lambda)^n / n!}$$

Eg. $\cancel{\lim_{N\to\infty}}$

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2}} P(0, 2\lambda) = e^{\lambda^2(\gamma+1) + \sum_{m=3}^{\infty} (-2\lambda)^m \frac{2^{m-1}-1}{2^{m-1}} \frac{\zeta(m-1)}{m}}$$

$$|\lambda| < \tfrac{1}{2}$$

Barnes' G-function

$$G(1+z) = (2\pi)^{z/2} e^{-[(1+\gamma)z^2 + z]/2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z + z^2/2n}$$

$G(1) = 1$

$G(z+1) = \Gamma(z) G(z)$ (*)

$$\log G(1+z) = (\log 2\pi - 1)\frac{z}{2} - (1+\gamma)\frac{z^2}{2} + \sum_{n=3}^{\infty} (-1)^{n-1} \zeta(n-1) \frac{z^n}{n}$$

$$\therefore \quad \lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \int_{U(N)} |Z(A,\theta)|^{2\lambda} d\mu(A) = f_U(\lambda)$$

$$f_U(\lambda) = \frac{G^2(1+\lambda)}{G(1+2\lambda)}$$

$\therefore$ using $(*)$

$$f_U(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

$$f_U(1) = 1, \quad f_U(2) = \frac{1}{12}, \quad f_U(3) = \frac{42}{9!}, \quad f_U(4) = \frac{24024}{16!} \quad \text{etc}$$

<u>conjecture</u> (K-S 1998)

$$\boxed{f_\zeta(\lambda) = f_U(\lambda) \qquad \operatorname{Re}\lambda > -\tfrac{1}{2}}$$

<u>Value distribution of $|Z|$</u>

$$\int_{U(N)} \delta(|Z(A,\theta)| - \omega)\, d\mu(A) = \rho_U(\omega, N).$$

obviously $\displaystyle \int_{U(N)} |Z|^t d\mu(A) = \int_0^\infty \rho(\omega,N)\,\omega^t\, d\omega$

$$\therefore \quad \rho(\omega,N) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+t)}{(\Gamma(j+\frac{t}{2}))^2} \frac{1}{\omega^{t+1}}\, dt$$

$N \to \infty$: st. phase

$w \to 0$: $\oint$ Cauchy (poles at -ve integers).