Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Derivatives of the Riemann zeta function

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RECENT PERSPECTIVES IN RANDOM MATRIX THEORY AND NUMBER THEORY

LECTURE 1: DERIVATIVES OF THE RIEMANN ZETA FUNCTION

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Abstract. Keating and Snaith used the characteristic polynomial of a random unitary matrix theory to model the moments of the zeta function. The same philosophy can be used to model the discrete moments of the derivative of the Riemann zeta function, evaluated at the non-trivial zeros.

The purpose of this lecture is to give a summary of how random matrix theory has been used to model the derivative of the Riemann zeta function. No proofs will be given, but references to the original research will be provided.

1. Moments of the zeta function

Let $\zeta(s)$ be the Riemann zeta function, and, assuming the Riemann Hypothesis, denote its non-trivial zeros by $\frac{1}{2}+i\gamma_n$, with $0<\gamma_1\leq\gamma_2\leq\ldots$ (see [16] for more background on the zeta function). Let $N(T)$ be the number of zeros with $0<\gamma_n\leq T$. We have

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T). \tag{1}$$

Moments of the Riemann zeta function have long been of interest to number theorists. Using random matrix theory, Keating and Snaith [12] created a conjecture for the asymptotic behaviour of the $2k^{\text{th}}$ moment:

Conjecture 1. *If $\Re e(k)>-1/2$, then*

$$I_k(T):=\frac{1}{T}\int_0^T\left|\zeta(\tfrac{1}{2}+it)\right|^{2k}\,dt \tag{2}$$

$$\sim\frac{G^2(k+1)}{G(2k+1)}a(k)\left(\log\frac{T}{2\pi}\right)^{k^2} \tag{3}$$

where G is the Barnes G-function, and

$$a(k)=\prod_{p\text{ prime}}\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2p^{-m}. \tag{4}$$

This conjecture agrees with all the known results on the zeta function, recorded in table 1.

Keating and Snaith obtained their conjecture by studying the characteristic polynomial of an $N\times N$ unitary matrix chosen with Haar measure, when

$$N=\log\frac{T}{2\pi}, \tag{5}$$

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$I_k(T)$	Result by	Theorem or conjecture
$I_1(T) \sim \log T$	Hardy and Littlewood [7]	Theorem
$I_2(T) \sim \frac{1}{2\pi^2}(\log T)^4$	Ingham [11]	Theorem
$I_3(T) \sim \frac{42}{9!}a(3)(\log T)^9$	Conrey and Ghosh [1]	Conjecture
$I_4(T) \sim \frac{24024}{16!}a(4)(\log T)^{16}$	Conrey and Gonek [2]	Conjecture

TABLE 1. Previously known moments of the zeta function

Setting

$$Z_U(\theta) := \det(I - e^{-i\theta}U) \tag{6}$$

$$= \prod_{n=1}^{N}\left(1 - e^{i(\theta_n - \theta)}\right), \tag{7}$$

then letting $\mathbb{E}_N$ denote expectation with respect to Haar measure on the group of $N \times N$ unitary matrices [17], they found

$$\mathbb{E}_N\left\{|Z_U(\theta)|^{2k}\right\} := \frac{1}{N!(2\pi)^N}\int_{[-\pi,\pi]^N}\prod_{1\le m<n\le N}\left|e^{i\theta_n} - e^{i\theta_m}\right|^2\prod_{n=1}^{N}\left|1 - e^{i\theta_n}e^{-i\theta}\right|^{2k}\,d\theta_n \tag{8}$$

$$= \prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(j+2k)}{\Gamma(j+k)^2} \tag{9}$$

$$\sim \frac{G^2(k+1)}{G(2k+1)}N^{k^2} \tag{10}$$

as $N \to \infty$ for fixed k subject to $\Re e(k) > -1/2$, where G is the Barnes G–function. This result was found by essentially evaluating the integral over Haar measure, (8), using a form of Selberg's integral (see chapter 17 of [13] for example).

Note that, after setting $N = \log(T/2\pi)$ the only difference between the moment of $|Z_U(0)|^{2k}$ and $|\zeta(\frac{1}{2}+it)|^{2k}$ (given in (10) and (3) respectively) is the factor $a(k)$, defined in (4). This is a zeta-function specific factor, and cannot be predicted by random matrix theory alone.

Remark. If k is a non-negative integer, then

$$\frac{G^2(k+1)}{G(2k+1)} = \prod_{j=0}^{k-1}\frac{j!}{(j+k)!} \tag{11}$$

2. Extremely small values of the zeta function

The Keating-Snaith conjecture allows us to predict how likely extremely small values of the zeta function are.

Since

$$\frac{1}{2\pi i}\int_{(1/2)}\frac{X^s}{|\zeta|^s}\frac{ds}{s} = \begin{cases}1 & \text{if } |\zeta| < X\\ 0 & \text{if } |\zeta| > X\end{cases} \tag{12}$$

where $(1/2)$ denotes the vertical line $\Re e(s) = 1/2$, we see that

$$\frac{1}{T}\operatorname{meas}\left\{0 \le t \le T \,:\, |\zeta(\tfrac{1}{2}+it)| \le X\right\} = \frac{1}{2\pi i}\int_{(1/2)} X^s \frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{-s}\,dt\frac{ds}{s} \tag{13}$$

Inserting the Keating-Snaith conjecture into the inner-most integral, gives that as $T \to \infty$,

$$\frac{1}{T}\,\mathrm{meas}\left\{0 \le t \le T \;:\; |\zeta(\tfrac{1}{2}+\mathrm{i}t)| \le X\right\} \sim \frac{1}{2\pi\mathrm{i}}\int_{(1/2)} X^s \frac{G^2(1-s/2)}{G(1-s)} a(-s/2)(\log T)^{-s^2/4}\frac{\mathrm{d}s}{s} \tag{14}$$

and if $X \le (\log T)^{-(1/2+\epsilon)}$, then the integral is dominated by the closest pole, at $s=1$, and expanding around that point we have

$$\frac{1}{T}\,\mathrm{meas}\left\{0 \le t \le T \;:\; |\zeta(\tfrac{1}{2}+\mathrm{i}t)| \le X\right\} \sim XG^2(\tfrac{1}{2})a(-\tfrac{1}{2})(\log T)^{1/4} \tag{15}$$

(for details see either [12] or §3.5.1 and §3.6.2 of [9]).

However, as $X \to 0$, we also have

$$\mathrm{meas}\left\{0 \le t \le T \;:\; |\zeta(\tfrac{1}{2}+\mathrm{i}t)| \le X\right\} \sim 2X \sum_{0<\gamma_n\le T} \left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^{-1} \tag{16}$$

which leads to

$$\frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^{-1} \sim \pi G^2(\tfrac{1}{2})a(-\tfrac{1}{2})\left(\log\frac{T}{2\pi}\right)^{-3/4} \tag{17}$$

$$= G^2(\tfrac{3}{2})a(-\tfrac{1}{2})\left(\log\frac{T}{2\pi}\right)^{-3/4} \tag{18}$$

(the truth of this result is dependent on the Keating-Snaith conjecture).

It is natural then to ask:

Question. Can random matrix theory model

$$\frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^{2k} \;?$$

3. Moments of the derivative of the zeta function

The question has been answered in the affirmative by Hughes, Keating and O'Connell [10].

Conjecture 2. *If all the zeros of the zeta function are simple, then if* $\Re\mathrm{e}(k) > -3/2$,

$$J_k(T) := \frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^{2k} \tag{19}$$

$$\sim \frac{G^2(k+2)}{G(2k+3)}a(k)\left(\log\frac{T}{2\pi}\right)^{k(k+2)} \tag{20}$$

where $a(k)$ *is given by* (4)*, and* $G(\cdot)$ *is the Barnes* G*–function.*

This result was also obtained from a study of the characteristic polynomial, for in [10] it was proven that if $\Re\mathrm{e}(k) > -3/2$ then as $N \to \infty$,

$$\mathbb{E}_N\left\{\frac{1}{N}\sum_{n=1}^{N}|Z_U'(\theta_n)|^{2k}\right\} = \frac{G^2(k+2)}{G(2k+3)}\frac{G(N+2k+2)G(N)}{N\,G^2(N+k+1)} \tag{21}$$

$$\sim \frac{G^2(k+2)}{G(2k+3)}N^{k(k+2)} \text{ as } N \to \infty. \tag{22}$$

This result was proven by factoring the Weyl density for Haar measure over the unitary group in a way that reduced the calculation to a integral of the form (8).

The existence of a pole at $k = -3/2$ in (21) means that $\mathbb{E}_N\left\{|Z_U'(\theta_N)|^{2k}\right\}$ diverges (for any $N \ge 2$) when $\Re\mathrm{e}(k) \le -3/2$.

3.1. **Discussion on the 'pole' at** $k = -3/2$. Due to the divergence of the random matrix average, conjecture 2 is restricted to $2\Re(k) > -3$. In this section, we argue that this restriction is necessary. First off, we should note that if there exists a multiple zero of the zeta function, then $J_k(T)$ diverges for any $k < 0$, so we assume all the zeros are simple (and up to an exceptional set of measure zero, this is the case for characteristic polynomials of random unitary matrices).

For k negative, but $|k|$ large, the sum over zeros of the zeta function may be dominated by the few points where $|\zeta'(1/2+i\gamma_n)|$ is close to zero. These points are expected to be where two zeros lie very close together (an occurrence of Lehmer's phenomena).

Gonek [6] defined

$$\Theta = \inf\left\{\theta : |\zeta'\left(\tfrac{1}{2}+i\gamma_n\right)|^{-1} = O\left(|\gamma_n|^\theta\right)\ \forall n\right\}. \tag{23}$$

If Θ is finite, then for any $\epsilon > 0$, there exists an infinite subsequence of the $\{\gamma_n\}$, such that

$$|\zeta'\left(\tfrac{1}{2}+i\gamma_n\right)|^{-1} > |\gamma_n|^{\Theta-\epsilon}. \tag{24}$$

Choosing a γ from this subsequence and setting $T=\gamma$, we have, for $k<0$,

$$J_k(T) > \frac{1}{N(T)}|\zeta'\left(\tfrac{1}{2}+i\gamma\right)|^{2k} \tag{25}$$

$$> \frac{2\pi}{T\log T}T^{-2k(\Theta-\epsilon)}, \tag{26}$$

If $\Theta > 0$, then

$$(\log T)^{k(k+2)} = o\left(\frac{2\pi}{T\log T}T^{-2k(\Theta-\epsilon)}\right) \tag{27}$$

when

$$2k < -\frac{1}{\Theta}, \tag{28}$$

implying that the conjecture is massively too small for $2k < -\frac{1}{\Theta}$.

Montgomery's pair correlation conjecture [14] suggests that

$$\liminf_{n\to\infty} \frac{\log(\gamma_{n+1}-\gamma_n)}{\log\gamma_n} = -\frac{1}{3} \tag{29}$$

which in turn suggests that $\Theta \geq 1/3$, and this gives rise to a number theory explanation of the "pole" at $k=-3/2$ in the conjecture.

3.2. **Comparison with previously known results.** Conjecture 2 is found to agree with all previously known results, when $k = -1, -1/2, 1, 2$.

First off, it agrees the conjecture made when $k=-1/2$ in (17), though this is not surprising since that result itself came from a random matrix calculation.

It also agrees with a conjecture of Gonek [5], that if all the zeros of the zeta function are simple, then

$$J_{-1}(T) \sim \frac{6}{\pi^2}(\log T)^{-1} \tag{30}$$

In an earlier paper, Gonek [4] showed that if the Riemann Hypothesis is true then

$$\frac{1}{N(T)}\sum_{0<\gamma_n\leq T}\left|\zeta\left(\tfrac{1}{2}+i(\gamma_n+\alpha/L)\right)\right|^2 \sim \left(1-\left(\frac{\sin(\pi\alpha)}{\pi\alpha}\right)^2\right)\log\frac{T}{2\pi} \tag{31}$$

uniformly in α for $|\alpha| \leq L/2$, where $L = \frac{1}{2\pi}\log T$. The coefficient of α^2 gives the second moment of $|\zeta'(\frac{1}{2}+i\gamma)|$ — that is, Gonek proves (subject to RH)

$$J_1(T) \sim \frac{1}{12}(\log T)^3 \tag{32}$$

A similar result is known for the fourth moment ($k=2$), though not in a large enough range to prove the result we want. Conrey, Ghosh and Gonek [3] prove the following theorem:

Theorem 3. *Assume GRH and let $A(s)=\sum_{n\le x} n^{-s}$ where $x=\left(\frac{T}{2\pi}\right)^{\eta}$ for some $\eta\in(0,\frac{1}{2})$. Then,*

$$\frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta A\left(\tfrac{1}{2}+\mathrm{i}(\gamma_n+\alpha/L)\right)\right|^2 \sim \frac{6}{\pi^2}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}(2\pi\alpha)^{2j+2}}{(2j+5)!}\times$$
$$\times\left(-\eta^2+\tfrac{1}{3}(2j+5)\eta^3-\frac{2j+5}{j+3}\eta^{2j+6}+\eta^{2j+7}+\eta^2(1-\eta)^{2j+5}\right)\left(\log\frac{T}{2\pi}\right)^4 \quad (33)$$

uniformly for bounded α.

Putting $\eta=1$ in the above (which, as it stands, is not allowed under the conditions of the theorem) then $A(\frac{1}{2}+\mathrm{i}t)=\zeta(\frac{1}{2}+\mathrm{i}t)+O(t^{-1/2})$, and we have

$$\frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta^2\left(\tfrac{1}{2}+\mathrm{i}(\gamma_n+\alpha/L)\right)\right|^2 \sim \frac{4}{\pi^2}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(2\pi\alpha)^{2j+2}}{(2j+6)!}(2j^2+5j)\left(\log\frac{T}{2\pi}\right)^4 \quad (34)$$

which implies that (by evaluating the coefficient of α^4)

$$\frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta'\left(\tfrac{1}{2}+\mathrm{i}\gamma_n\right)\right|^4 \sim \frac{1}{1440\pi^2}\left(\log\frac{T}{2\pi}\right)^8. \quad (35)$$

This is the same answer that one gets from putting $k=2$ into conjecture 2.

Though this is not a theorem, due to Theorem 3 not being proven in the case $\eta=1$, it is known that this result is of the right order of magnitude, for Nathan Ng [15] showed

$$\frac{(\sqrt{97}-\sqrt{61})^2}{30240\pi^2}\left(\log\frac{T}{2\pi}\right)^8 \le \frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta'(\tfrac{1}{2}+\mathrm{i}\gamma_n)\right|^4 \le \frac{(\sqrt{97}+\sqrt{61})^2}{30240\pi^2}\left(\log\frac{T}{2\pi}\right)^8 \quad (36)$$

4. Another discrete moment

It is striking that conjecture 1 and conjecture 2 have very similar form. Indeed, they have been unified as special cases of one result [8].

Conjecture 4. *For $\Re e(k)>-1/2$,*

$$M_k(T;\alpha):=\frac{1}{N(T)}\sum_{0<\gamma_n\le T}\left|\zeta\left(\tfrac{1}{2}+\mathrm{i}\left(\gamma_n+\frac{2\pi\alpha}{\log(T/2\pi)}\right)\right)\right|^{2k} \quad (37)$$

$$\sim\frac{G^2(k+1)}{G(2k+1)}a(k)F_k(2\pi\alpha)\left(\log\frac{T}{2\pi}\right)^{k^2} \quad (38)$$

uniformly for $|\alpha|<1$, where $a(k)$ is given by (4), G is the Barnes G–function, and

$$F_k(2x)=\tfrac{1}{2}\pi x J_{k+1/2}(x)^2+\tfrac{1}{2}\pi x J_{k-1/2}(x)^2-k\pi J_{k+1/2}(x)J_{k-1/2}(x) \quad (39)$$

where $J_\nu(x)$ is the ν^{th} Bessel function of the first kind.

As in all the previous cases, the conjecture is deduced from showing that

$$\mathbb{E}_N\left\{\left|Z_U\left(\theta_1+\frac{2\pi\alpha}{N}\right)\right|^{2k}\right\}\sim\frac{G^2(k+1)}{G(2k+1)}F_k(2\pi\alpha)N^{k^2} \quad (40)$$

Remark. When k is an integer, the Bessel functions can be simplified:

$$F_1(2x) = \frac{x^2 - \sin^2(x)}{x^2} \tag{41}$$

$$F_2(2x) = \frac{x^4 - 3x^2 + 3x\sin(2x) + (2x^2 - 3)\sin^2(x)}{x^4} \tag{42}$$

$$F_3(2x) = \frac{x^6 - 3x^4 - 45x^2 + (-12x^3 + 45x)\sin(2x) + (-3x^4 + 72x^2 - 45)\sin^2(x)}{x^6} \tag{43}$$

This conjecture is found to agree with the known result of Gonek [4] in the case $k = 1$, and the extension of a theorem due to Conrey, Ghosh and Gonek [3], mentioned above in (34).

Taking the limit as $\alpha \to 0$ one may recover conjecture 2, and the limit $\alpha \to \infty$ leads to a variant of conjecture 1. (For details see [8]).

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