Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Computational methods for L-functions I

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Computational Methods for L functions

Michael Rubinstein

Lecture

Emphasis on analytic computations as opposed to algebraic computations (i.e we'll assume that we are handed basic data for the L-function, such as functional eqn, Dirichlet coefficients).

Elementary techniques

- Summation / Integration by parts
- Euler Maclaurin Summation
- Mobius Inversion

Slightly less elementary techniques from analysis

- Poisson summation
- Cauchy's residue thm
- asymptotic methods
- fft

Integration by parts

$$\int u\ t\ v(t)\,dt \quad u(t)v\ t) \quad \int u\ t\ v(t)\,dt$$

Summation by parts

$$\sum_{<n\leq x} f(n)g(n) \quad \Big(\sum_{1\leq n<x} f(n)\Big)g(x) \quad \int^{x}\Big(\sum_{<n<t} f(n)\Big)g(t)\,d$$

(assume g continuous $[1,x]$

can start lier

ex/

$$\pi(x \quad \sum_{<x} \quad - \sum_{\leq x} \log p \quad \log p$$

$$- \Big(\sum_{p\ x} \log p\Big) \log x + \int^{x} \Big(\sum_{p<t} \log\Big) \frac{dt}{t(\log t)^2}$$

If $\sum_{p<x} \log p \sim x$ then $\pi(x) \sim \frac{x}{\log x}$

Similarly

$$\sum_{p\ x} \log p \quad \Big(\sum_{p\ x}\Big) \log x - \int^{x} \Big(\sum_{p<} 1\Big) \frac{dt}{t}$$

So f $\pi(x) \sim \frac{x}{\log x}$ then $\sum_{p<x} \log p \quad x$

Euler Maclaurin Summation (followed by repeated integration by parts)

$$\sum_{1<n<x} g \quad) \quad \lfloor x \rfloor g(x) \quad \int_0^x \lfloor t \rfloor g\, t \; dt$$

starting + 0 leads to clean for

$a\ b \in \mathbb{Z}$

$$\sum_{a<n\le b} g(n) \quad (b \quad) g(b) \quad \int^b (\lfloor t \quad a \ \ g(t \ dt$$

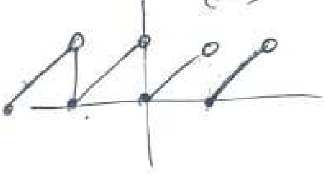
$$b g(b) \quad a g(a \quad \int_a^b \lfloor t \rfloor g(t \ dt$$

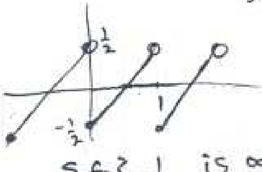
$\lfloor t \rfloor \quad t \quad \{t\}$ $\quad \{t\}$ the fraction part f t

$$\sum_{\ <n\ \ b} g(n) - \int_a^b g(t)\,dt + \underbrace{\int_a^b \{t\} g(t)\,dt}$$

correction that we get from replacing the sum with an integral

$\{t\}$





$\{t\}-\frac{1}{2}$ is odd nicer (ex constant term in fourier series is zero)

$$\sum_{<n\ b} g(n) \quad \int_a^b g(t)\,dt + \frac{1}{2}\left(g(b)\ g(a)\right)$$

$$+ \int_a^b \left(\{t\}\ \frac{1}{2}\right) g\ t\ dt$$

Bernoul p lynomia s

defn $B\ (t)$

$B_k'(t)\quad k B_k\quad t)$ determin s B_k t n terms f $B_k\ (t)$ up to constant

$\int_0^1 B_k\ t)\,dt \quad 0 \quad k \geq$ fixes the constant

k	$B_k(t)$
0	1
	$t\ 1/2$
2	$t\ \ t + 1/6$
3	$t\ \ 3/2\,t^2 + 1/2\,t$
4	$t\ \ -2t^3 + t^2\ \ 30$
5	$t\ \ 5/2\,t^4 + 5\ 3t\ \ 1/6\,t$

Let B_k $B_k(0$ denote the constant ter of $B_k(t)$

Properties of Bernoul i polynom ls

) B_k t) $- \sum^{k} \binom{k}{} B_k \; t^m \quad k \geq 0$

2) $\frac{z e^{zt}}{e^{z}} \quad \sum_{0}^{\infty} B_k(t) \frac{z^k}{k} \quad z < 2\pi$

3) $B_k(t) \quad (-1)^k B_k(1-t \quad k \geq 0$

4) $\frac{B_k \quad t) \quad B_{k+1} \; t}{k+1} \quad t^k \quad k \geq 0$

5) $B_k \quad) \quad \left. \begin{array}{ll} (-)^k B_k(1) & k \geq 0 \\ 0 & k \text{ odd} \geq 3 \\ 1/2 & k \\ B_k(0) & k \text{ even} \end{array} \right\}$ $B_k(1)$ $B_k(0)$ unless $k =$

6) $0 - \sum_{m=0}^{k} \binom{k}{} B_m \quad k \geq 2$

7) $B_1(\{t\}) \quad \frac{}{\pi} \sum^{\infty} \frac{\sin(2\pi m t)}{} , \quad t \notin \mathbb{Z}$

$k \geq 2$ $B_k \{t\}) \; -k \sum_{m \neq 0} \frac{e^{\pi i m t}}{(2\pi i m)^k}$ is continous

$k > 2$

$$B_k(\{t\}) \quad \frac{k}{(2\pi i)^k} \sum_{n \neq 0} \frac{e^{2\pi i \; t}}{n^k}$$

Take t 0, $k-2m$ eve

$$B_{2m} - \frac{(-1)^{\;+1} e}{(2\pi)^{2m}} 2\zeta \; 2 \;)$$

$$\zeta \; 2m) - \frac{(2\pi)^{2m} (-)^{m+1} B_2}{2 \;\; 2m)}$$

$$\int_a^b \left(\{t\} - \frac{1}{2}\right) g(t)\, dt \quad \int_a^b B\left(\{t\}\right) g(t)\, dt$$

Break up $\int_a^b \quad \int_a + \int^{a+1} \quad \int^b$

$$\sum_{n \quad a}^{b-} \frac{B_2(1)\, g'(m+1) - B_2(0)\, g'(m)}{2} \quad \int_a^b \frac{B_2(\{t\})}{2} g''(t)\, dt$$

$B_2 \quad B_2\, 0) \quad 1/$

telescoping sum $=$ $\dfrac{B_2}{2}\left(g(b) \quad g\; a\right) \quad \int_a^b \frac{B_2(\{t\})}{2} g''(t)\, dt$

Repeat use $B_k(1) \quad B_k(0) \quad f\; k > 2$

Euler-Maclaurin formula

$$\boxed{\sum_{a \quad n \quad b} g(n) \quad \int_a^b g(t)\,dt + \sum_{k} \frac{(-1)^k}{k!} B_k\left(g^{(k-)}(\;) \quad g^{(k-1)}(a\right) + \frac{(-1)^K}{K} \int_a^b B_K(\{t\})\, g^{(K}(t)\, dt}$$

Applications

$$\sum_{n}^{N} n^r$$

take tole $f \to 0$

$g(t) - t^r \quad g^{(m)}(N) - g^{(m}(0) \quad \overbrace{r(r-1)\ldots(r-m+1)N^{r}}^{\text{take tole}} \quad m\ r$

$0 \quad m \geq r$

$g^{(r+1)}(t \quad 0$

$$\sum^{N} n^r - \int_0^N t^r\,dt + \sum_k^r (\ 1)^k \frac{B_k}{k} r(r\ ).\ (r-k\ )N^{\ k}$$

$$-\int_0^N t^r dt + \sum_k^r (-\ ^k \frac{B_k}{r-k+1}\binom{r}{k} N^{r+1\ k}$$

$$-\int_0^N \underbrace{\sum_{k=0}^r (-1\ ^k B_k \binom{r}{k} t^{r\cdot k}}_{(-1)^r B_r(\ t)\quad B_r(t+\ )} dt \qquad \int_0^N B_r\ t+1\ dt$$

$$-\frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1} \quad , \quad \frac{B_{r+1}(N+1) - B_{r+}}{r+}$$

Can verify directly

$$\frac{B_{r+1}(n+1) - B_{r+1}(n)}{r+} \quad n^r \quad r > 0$$

Σ into N l.h.s telescopes

Application 2

$\zeta(s \quad \sum^{\infty} \frac{}{n^s} \quad \mathrm{Re}\, s >$

Consider

$\sum^{N} n$

$g(t \quad t^{s}, \quad g^{(m)}(t \quad (- \,)^m \underbrace{s\cdot(s+1)\cdots(s+m-\,)}_{\text{take to be if } 0} t^{s}$

$\sum_{1}^{N} n^{s} \quad + \sum_{2 \leftarrow \text{to avoid pole at } t\ 0}^{N} n^{s}$

$1 + \underbrace{\int_{1}^{N} t^{-s}\,dt}_{\frac{N^{-s+1}}{s} \quad 1-s} \qquad \sum_{k=1}^{K} \frac{B_k}{k} \binom{s+k\ \ 2}{k} \left(N^{s-k} \quad \right)$

$\left(\begin{smallmatrix} s+K \\ K \end{smallmatrix}\right) \int^{N} B_K(\{t\})\, t^{s-K}\, dt$

Take limit as $N \to \infty$, and since $\mathrm{Re}\, s > 1$

$$\boxed{\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{2}^{K} \binom{s+k-2}{k-1} \frac{B_k}{k} - \binom{s+K-1}{K} \int_{1}^{\infty} B_K(\{t\})\, t^{-s-K}\, dt}$$

While we started with $\mathrm{Re}\, s > 1$, r.h.s is meromorphic for $\mathrm{Re}\, s > -K+1$, so gives the meromorphic continuation f $\zeta($ i this region

Take s 2 $\mathbb{K}$ $\mathbb{K} > 2$

$$\zeta(2\ \mathbb{K}) \qquad \mathbb{K}\ 1 \sum_{k}^{\mathbb{K}\ 1} (-1\ ^{k} \binom{\mathbb{K}\ 1}{k} B_k$$

$$\frac{-(\ )^{\mathbb{K}} B_{\mathbb{K}}}{\mathbb{K}\ 1}$$

$\zeta(\ 2\ )$ $\quad B_{2m}/2m$ $\quad m\ 1\ 2$

$\zeta(\ 2m) - 0$ $\qquad m\ \ 2, 3, \ldots$

$\zeta(0$ $\qquad 2$

Apply funct eqn $\pi^{s/2}\Gamma\ \frac{s}{})\zeta(\ )\ \ \pi^{\frac{(1-s)}{}}\Gamma\ \ \frac{-s}{2})\zeta(1-s)$

and $\Gamma(\ _2)\ \pi\ 2\quad \frac{1}{2}\ \frac{3}{2}\ \frac{-5}{2}\quad \frac{-(2m-1}{2}\ \Gamma(\frac{}{-2}\quad)$

Gives

$$\zeta(2m) - \frac{(-)^{\ +}(2\pi)^{2}}{2\cdot(2m)!} B_{2m} \qquad m \geq$$

Since $\zeta(2m) \sim$ as $m \to \infty$

$$B_{2m} \sim \frac{(-)^{m+}(2m)\ \ 2}{(2\pi)^{2m}}$$

For computations it's better to do the following

$$\sum n \quad \underbrace{\sum^{N} n^{s}}_{\text{keep as is}} \quad \underbrace{\sum_{N} n}_{\text{apply Euler-Mac}}$$

$$\sum_{n>N} n^{s} \qquad \frac{N}{s} + \sum^{R} \binom{s+k-2}{k} \frac{B_k}{k} N^{s-k+1}$$

$$\binom{s+R}{R} \int_{N}^{\infty} B_R(\{t\}) t^{s-R}\, dt$$

again for $\mathrm{Re}\, s > R+$

Estimate size of

$$\binom{s+R-1}{R} \int_{N}^{\infty} B_R(\{t\}) t^{s-R}\, dt$$

Convenient to take $R = 2M$ even

$k \geq 2$

$$\sim \left(\{t\}\right) - k \sum_{m \neq 0} \frac{e^{2\pi i \, t}}{(2\pi i m)^k}$$

$$B_k(\{t\}) < \frac{k}{(2\pi)^k} 2 \zeta(k)$$

if k is even

$m \geq 1$

$$B_{2m}(\{t\})\Big| \leq |B_{2m}$$

$$\left| \binom{s+2m-}{2m} \int_N^{\infty} B_{2m}(\{t\}) t^{\quad m} dt \right|$$

$$< \left| \binom{s+2m}{2m} \right| B_{2m}| \frac{N^{\sigma \ 2m}}{\sigma+2m} \qquad s = \sigma + i\tau, \quad \sigma > 2m+$$

$$= \frac{s+2m \ 1}{\sigma \ 2m} \left| \binom{s+2m \ 2}{2m} B_{2m} \right| N^{\sigma \ 2m}$$

$$\frac{s+2m-1}{\sigma+2m}$$ ast term taken

$$|B_{2m}| \quad 2\frac{(2m)}{(2\pi)^{2m}}\zeta(2m)$$

Remainder is $<$

$$\frac{\zeta(2m)}{\pi}\frac{1}{N^{\sigma}}\frac{|s+2m}{\sigma+2m}\prod_{0}^{2m-2}\frac{s+j|}{2\pi N}$$

We start to win wh $2\pi N$ is bigger

than s $|s+ \qquad +2m$ 2

2 parameters M N and also # digits

accuracy we want

For example e can take

$\sim$ # digits (we need the 1st bit extra to kill all $O(\log|s|)$ extra)

$$2\pi N \geq 0 \max_{j\ 0}^{2m-2} s\ j$$

Computing products sums over primes

ex1

$$2 \prod_{p>2} \frac{p(p-2}{(p-1)^2} \quad \text{twin prime constant}$$

ex2

conj ks

$$M_k(T) \quad T\int_0^T \zeta\ 1/2\ ct)\ ^k dt \quad \frac{a_k g_k}{(k\ )!} (\log T)^{k^2}$$

$$a_k \quad \prod_p \quad p)^k \sum_n^\infty \binom{n+k-1}{n}^2 p^{\ n}$$

$$\frac{g_k}{(k^2)} \quad \prod_j^k \frac{j!}{(j+k}$$

ex2b

conj pcks

$$M_k\ T) \quad \sum_r^{k^2} c_r\ k)(\log T)^{k^2\ r} + O\ \ T^{1/2\ \epsilon})$$

where $C_0(k)$ $a_k g_k/(k\ )$

$C\ \ k)$ $C_0\ k)$ (rational function $n\ k$ generalized Euler constants and sums over primes involving $\log p$ and $_2F\ \ k,k,\ \ p^{-})$ and its derivatives

Numer cal evaluation of these constants boils down to being able to compute $\sum_p \frac{(\log p)^r}{p^m}$ $r=0,1,2,3,$

The mobius function $\mu(n)$:

n:	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(n)$:	1	-1	-1	0	-1	1	-1	0	0	1	-1	0

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^{\square \text{ prime factors of } n} & \text{if } n \text{ is } \square \text{ free.} \end{cases}$$

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum \frac{\mu(n)}{n^s}$$

$$1 = \frac{1}{\zeta(s)} \cdot \zeta(s) = \sum \frac{\mu(n)}{n^s} \sum \frac{1}{m^s}$$

$$= \sum_{m,n} \frac{\mu(n)}{(n \cdot m)^s} = \sum_{r=1}^{\infty} \frac{1}{r^s} \sum_{n|r} \mu(n)$$

So $\displaystyle\sum_{n|r} \mu(n) = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{otherwise} \end{cases}$

rify rectly $\displaystyle\prod_{p|r} (1-1) = \begin{cases} 1 & r = 1 \\ 0 & \text{otherwise} \end{cases}$

$\zeta\ )\ \prod_p (\quad \frac{1}{p^s})^{1}\quad \mathrm{Re} >$

$\log \zeta(s)\quad \sum \frac{1}{s}\quad \frac{1}{2}\sum \frac{1}{p^{2s}} + \frac{1}{3}\sum \frac{1}{p^{3s}}$

Let

$h(s\quad \sum \ p^s$

$\log \zeta\ s)\quad h(s\quad {}_2 h(2s) + {}_3 h\ 3s) + {}_4 h(4s) + {}_5 h(5s) + \frac{1}{6} h\ 6s) +$

$-\frac{1}{2}\log \zeta(2s)\quad \frac{}{2} h(\ s)\quad {}_4 h(4s)$ subtracted twice $\frac{}{6} h(6s) -$

$\frac{1}{3}\log \zeta(3s)\quad \frac{}{3} h(3s)\quad \frac{}{6} h(6s) -$

${}_5 \log \zeta(5s\quad {}_5 h(5s)$

$+ {}_6 \log \zeta(6s)\quad + {}_6 h(6s) +$

etc

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m} \log \zeta(ms)\quad \sum^{\infty} \frac{\mu(m)}{m} \sum_{n=}^{\infty} \frac{1}{n} h\ mns)$$

$$, = \sum_{r=}^{\infty} \frac{h(rs)}{r} \underbrace{\sum_{m|r} \mu\ m)}_{\substack{r= \\ \text{therwise}}}\quad h(s)$$

$$\sum_p \frac{}{p^s} \qquad \sum^{\infty} \frac{\mu(m)}{m} \log \zeta(ms) \qquad \mathrm{Re} >$$

Notice

$$\zeta(\ \ s) \quad + \frac{}{2^{ms}} + \frac{}{3^{ms}} + \quad \longrightarrow$$ exponentially fast as $m \to \infty$

$\log \zeta(ms) \to 0$ exponentially fast as $m \to \infty$

Number terms needed on r.h.s proportional to the number of digits accuracy we want

By differentiating r time we als g t

$$\sum_p \frac{(\log p)^r}{p^s} \quad (-\ )^r \sum^{\infty} \frac{\mu(m)}{m} \left(\log \zeta(ms\ \ )\right)^{r)}$$

ex/

$$\prod_{p>2} \frac{p(p-2)}{(p\ \ 1)^2} \quad \prod_{p>2} \frac{\ \ \frac{2}{p})}{(\ \ \frac{1}{p})^2} - 660\ 6\ 81584686957$$

$$- \exp\left(\sum_{p>2} \log(\ \ \tfrac{2}{p})\ \ 2\log(1\ \ \tfrac{1}{p}) \right)$$

Let $f(p \quad \log(\ \ \frac{2}{p})\ \ 2\log\ 1\ \ \frac{1}{p})$

$$f(p \quad \sum_{m}^{\infty} \frac{2\ \ \ \ 2}{m\, p}$$

$$\sum_{p>2} f\ p) \quad - \sum^{\infty} \frac{2\ \ \ 2}{m} \left(h\ \ \ )\ \ \ \ 2 \right)$$

we ar taking $p>2$

$$h(m) - \frac{1}{2} \qquad \frac{1}{3}$$

sum converges exponentially fast

we can get faster convergen e

$$\sum_{p>2} f(p) \quad \underbrace{\sum_{2<p\le P} f(p}_{\text{here the terms}} + \underbrace{\sum_{p>P} f(p)}_{\text{here the serie for fi but ed fewer terms p bigge}}$$

Poisson Summation as a tool for numerical integration

$$\sum f(n) = \sum \hat{f}(n) \quad , \quad f \text{ smooth, rapidly decreasing.}$$

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i y t} dt$$

$$\Delta \sum f(n\Delta) = \sum \hat{f}\left(\frac{n}{\Delta}\right)$$

$$= \underbrace{\hat{f}(0)}_{\int_{-\infty}^{\infty} f(t)dt} + \sum_{n \neq 0} \hat{f}\left(\frac{n}{\Delta}\right)$$

So

$$\int_{-\infty}^{\infty} f(t)dt - \Delta \sum f(n\Delta) = \sum_{n \neq 0} \hat{f}\left(\frac{n}{\Delta}\right)$$

tells you how closely the Riemann sum $\Delta \sum f(n\Delta)$ approximates $\int_{-\infty}^{\infty} f(t)dt$

main point: if $\hat{f}$ is rapidly decreasing then we get enormous accuracy from the Riemann sum even with Δ not too small.

ex

$$f(t) \quad e^{t/2}$$

$\hat{f}(y) \sqrt{2\pi} e^{2\pi y^2}$ so $\sum_{n \neq 0} \hat{f}\left(\frac{n}{\Delta}\right) - O\left(e^{\frac{2\pi^2}{\Delta^2}}\right.$

So

$$\underbrace{\int_{\infty}^{\infty} e^{t/2}\, dt}_{2\pi} \quad \Delta \sum_{\infty}^{\infty} e^{(n\Delta)^2/2} - O\left(e^{\frac{2\pi}{\Delta^2}}\right)$$

For example taking Δ $1/10$ we get

$$\Delta \sum^{\infty} e^{(n\Delta)^2} \quad - \sqrt{2\pi} + \varepsilon$$

th $\varepsilon \approx 0^{857}$

W can truncate the ∞ sum roughly when

$$\frac{(n\Delta)^2}{2} > \frac{2\pi^2}{\Delta^2}$$

i.e. $n > \frac{2\pi}{\Delta^2}$

S about 628 terms $\left(\begin{smallmatrix}\text{combine}\\ \pm n\end{smallmatrix}\right)$ are needed to get about 857 decim places

Keating Snaith

$U_{N \times N}$ unitary

$Z(U, \theta) \prod_{n}^{N} (e \quad e^{n})$ char function of U e luated on unit circle

$$M_{U(N)}(2k\ ) - \frac{}{(2\pi)^N N} \int_0^{2\pi} \int_0^{2\pi} d\theta \quad d_N$$

$$\times \prod_{k j c} |e^{i\theta} e|^2 |Z u)|^{2k}$$

$\mathrm{Re}\, k > \ _2$

$$\prod^{N} \frac{\Gamma(j)\Gamma(j+2k)}{\Gamma^2(\ k)}$$

selberg integral

Given the moments of probability density function that is supported on $t > 0$

$$M\ r) \int_0^{\infty} p(t\ t^r dt$$

we can recover $p(t$ by taking nverse Mellin transform

$$p(t) \frac{1}{2\pi i t} \int M(r) t^{-r} dr$$

c to the right of poles f $M(r)$

n

One possibility is to shift line of integration to the left picking up residues.

In our example

$$M_{U(N)}(r) = \prod_{1}^{N} \frac{\Gamma(j)\Gamma(j+r)}{\Gamma^2(j+\frac{r}{2})}$$

we get poles at $r = -1, -2, -3,$

As N grows, residues become harder to compute Instead, one can evaluate the integral as a Riemann sum

$$\int_{c-i\infty}^{c+i\infty} \prod \frac{\Gamma(j+r)}{\Gamma^2(j+\frac{r}{2})} t^{-r} \, dr$$