Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Toeplitz determinants & connections to random matrices II

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Recall

$$T_n(\varphi) \quad P_n T(\varphi) P_n$$

$$T(\varphi) \quad H^2 \longrightarrow H^2$$

$$T(\varphi) f \qquad P(\varphi f)$$

$$T(\psi_- \varphi \psi_+)$$

$$= T(\psi_-) T(\varphi) T(\psi_+)$$

$$T(\varphi \psi) = T(\varphi) T(\psi) + H(\varphi) H(\tilde{\psi})$$

$$T(\varphi) T(\varphi^{-1}) = I + \text{trace class}$$

(for "nice" φ)

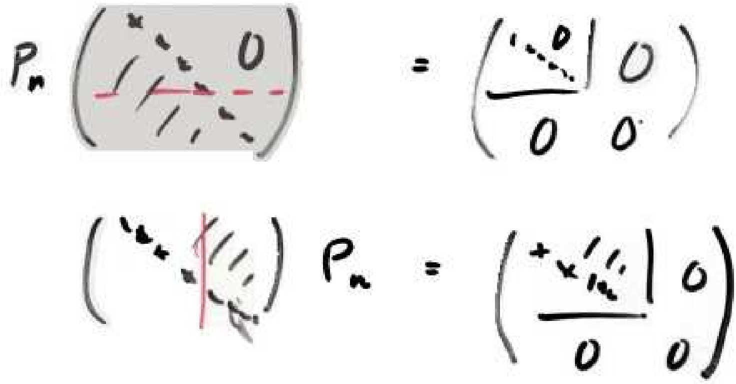
$$\det(T(\varphi) T(\varphi^{-1})) \quad \exp\left(\sum_1^\infty k s_k s_{-k}\right)$$

$$(\log \varphi)_k \quad s_k$$

Also notice

$$P_n T(\varphi_+) = P_n T(\varphi_+) P_n$$

$$T(\varphi_-) P_n = P_n T(\varphi_-) P_n$$



Theorem Geronimo Case
Borodin-Okounkov

Suppose φ satisfies the conditions of the previous theorem. Then

$$\det T_n(\varphi)$$

$$= G(\varphi)^n \, Z \, \det(I \quad K_n)$$

where

$$G(\varphi) = \exp\left(\frac{1}{2\pi}\int_{\pi}^{\pi} \log \varphi(e^{i\theta})\, d\theta\right)$$

$$Z = \det\left(T(\varphi)\, T(\varphi^{-1})\right)$$

$$K_n = Q_n \, H(\varphi/\varphi_+)\, H(\tilde{\varphi}_+/\tilde{\varphi}_-)\, Q_n$$

$$Q_n \quad I \quad P_n \quad \text{on } H^2$$

Pf $\quad P_n T(\varphi) P_n$

$$= P_n T(\varphi_- \ \varphi_+) P_n$$

$$= P_n T(\varphi \ ) T(\varphi_+) P_n$$

$$P_n T(\varphi_+) T(\varphi_+^{-1}) T(\varphi_-) T(\varphi_+) T(\varphi_-^{-1}) T(\varphi_-) P_n$$

$$P_n T(\varphi_+) P_n$$

$$\times \ P_n T(\varphi_+^{-1}) T(\varphi_-) T(\varphi_+) T(\varphi_-^{-1}) P_n$$

$$\times \ P_n T(\varphi_-) P_n$$

$$\det P_n T(\varphi_+) P_n \quad (\varphi_+)_0^n$$

$$\det P_n T(\varphi \ ) P_n = (\varphi \ )_0^n$$

$$(\varphi_+)_0^n \qquad e^{n(\log \varphi_+)_0}$$

$$(\varphi)_0^n \qquad e^{n(\log \varphi)_0}$$

$$(\varphi_+)_0^n \, (\varphi)_0^n = e^{n(\log \varphi)_0} = G^n$$

So we have $D_n(\varphi)$

$$= G^n \det P_n A P_n$$

where $A = T_{\varphi_+^{-1}} T_{\varphi_-} T_{\varphi_+} T_{\varphi_-^{-1}}$

$$A = I + K \leftarrow C$$

trace class

$$\det A = \det T_{\varphi_+} T_{\varphi_-} T_{\varphi_+} T_{\varphi_-^{-1}}$$

$$= \det T_{\varphi_-} T_{\varphi_+} T_{\varphi_-^{-1}} T_{\varphi_+^{-1}}$$

$$\det(T(\varphi)\, T(\varphi^{-1}))$$

Now we use a wonderful formula

$$\det P'AP' = \det A$$

$$\times \det Q' A^{-1} Q'$$

where $P' + Q' = I$, or

$$\det T_n(\varphi)$$

$$= G(\varphi)^n \det T(\varphi)T(\varphi^{-1})$$

$$\times \det Q_n A^{-1} Q_n$$

$$A^{-1} = T(\varphi_-)T(\varphi_+^{-1})T(\varphi_-^{-1})T(\varphi_+)$$

$$= T(\varphi_-/\varphi_+)T(\varphi_+/\varphi_-)$$

$$= I \quad H(\varphi_-/\varphi_+)H(\tilde{\varphi}_+/\varphi_-)$$

so

$$\det Q_n A^{-1} Q_n$$

$$= \det (Q_n (I - H(\varphi_- / \varphi_+) H(\widetilde{\varphi_+ / \varphi_-}) Q_n$$

$$= \det (Q_n - Q_n H(\varphi_- / \varphi_+) H(\widetilde{\varphi_+ / \varphi_-}) Q_n)$$

$$= \det (I - Q_n H(\varphi_- / \varphi_+) H(\widetilde{\varphi_+ / \varphi_-}) Q_n).$$

Cor Since $Q_n \to 0$ strongly strongly the last term $\longrightarrow \underline{1}$ as $n \to \infty$

so

$$D_n(\varphi) / G(\varphi)^n$$

$$\longrightarrow \underbrace{\det(T(\varphi)\, T(\varphi^{-1}))}_{\exp\left(\sum_1^\infty k s_k s_{-k}\right)}$$

as $n \to \infty$

Non-Smooth Symbols

or

Computing $\det T_n(\varphi)$ when φ does not satisfy

$$\sum_1^{\prime} k\,|\varphi_k|^2 < \infty$$

In 1968 Fisher + Hartwig considered a class of symbols of the form

$$\psi(e^{i\theta}) = \varphi(e^{i\theta}) \prod_{j=1}^{R} \varphi_{\alpha_j,\beta_j}(e^{i(\theta-\theta_j)})$$

$$\varphi_{\alpha,\beta} = (2 - 2\cos\theta)^{\alpha} e^{i\beta(\theta-\pi)}$$

$$\operatorname{Re}\alpha > -\tfrac{1}{2}$$

Here φ is a "nice" function, the term $(2 \quad 2\cos\theta)^{\alpha}$ may have a zero, be unbounded, $^{i\beta(\theta-\pi)}$ has a jump discontinuity if $\beta \notin \mathbb{Z}$

F-H conjectured that

$$D_n(\psi) \sim G(\varphi)^n \, n^{\Omega} \, E^*$$

where $\Omega \quad \sum_1^R (\alpha_j^2 \quad \beta_j^2)$

and E^* is a constant

The conjecture is now confirmed in many cases. Here is the constant

$E^*(\varphi)$

$$= E(\varphi) \prod_{j=}^{R} \varphi_+(e^{\theta_j})^{\alpha_j+\beta_j} \varphi_-(e^{\theta_j})^{\alpha_j-\beta_j}$$

$$\times \prod_{\leq s \neq r \leq R} (\quad e^{i(\theta_s \ \theta_r)})^{(\alpha_s+\beta_s)(\alpha_r-\beta_r)}$$

$$\times \prod_{j=}^{R} \frac{G(1+\alpha_j+\beta_j)G(\ +\alpha_j \ \beta_j)}{G(\ +2\alpha_j)}$$

$G(z)$ is the Barnes G-function

$$G(1+z) = \Gamma(z)G(z)$$

$$G(1+z) \quad (2\pi)^{z/2} \quad -(z+1)z/2 - \gamma z^2/2$$

$$\times \prod_{k=}^{\infty} (1+z/k)^k e^{-z+z^2/2k}$$

Known now to be ture

if $|\mathrm{Re}\,\alpha_j| < \frac{1}{2}$, $|\mathrm{Re}\,\beta_j| < \frac{1}{2}$

One easy case

$R = 1, \quad \alpha = 0, \quad \varphi = 1$

$$\psi(e^{i\theta}) = e^{\beta i(\theta \quad \pi)}$$

Matrix s Cauchy,

Conjecture conf rmed by

Fisher + Hartwig

Idea What can one say about

$$D_n(\varphi\psi) \Big/ D_n(\varphi)\, D_n(\psi)$$

when φ, ψ do not have common singularities?

Recall that

$$T(\varphi\psi) \quad T(\varphi)T(\psi) = H(\varphi)H(\tilde{\psi})$$

An analogue for finite matrices was noticed by H Widom

$$T_n(\varphi\psi) \quad T_n(\varphi)T_n(\psi)$$

$$- \; P_n H(\varphi)H(\tilde{\psi})P_n$$

$$+ \; W_n H(\tilde{\varphi})H(\psi)W_n$$

(For now, let α_2 0)

where

$$W_n(a_0, a_1, \ldots, a_{n-1}, a_n, \quad)$$

$$= (a_{n-1}, \quad , a_0, 0, 0, \ldots \quad)$$

$W_n \longrightarrow 0$ weakly

$$\langle W_n e_j, e_k \rangle = \delta_{n-1, k}$$

0 if n is large enough

Also $W_n^2 = P_n$

Let's go back to the identity to consider

$$T_n(\varphi\psi)$$

$$- \quad T_n(\varphi)T_n(\psi) \;+\; A_n + B_n$$

or

$$T_n(\varphi\psi)\, T_n(\psi)^{-1}\, T_n(\varphi)^{-1}$$

$$= I_n \;+\; A_n T_n(\psi)^{-1} T_n(\varphi)$$

$$+ \; B_n T_n(\psi)^{-1} T_n(\varphi)^{-1}$$

We nvestigate

$$K_n \quad P_n \underbrace{H(\psi)H(\tilde{\varphi})}\, P_n T_n(\psi)^{-1} T_n(\varphi)^{-1}$$

Is this trace
class

Lemma

Suppose $\phi, \psi \in L^\infty$ and there exists a smooth partition of unity f, g such that ϕf, ψg satisfy the condition

$$H(\phi f), \ H(\tilde{\psi g})$$

are trace class Then

$$H(\phi) H(\tilde{\psi})$$

is trace class

Pf $\quad H(\varphi)H(\tilde{\psi})$

$$= T(\varphi\psi) - T(\varphi)T(\psi)$$

$$= T(\varphi(f+g)\psi)$$

$$\quad - T(\varphi)T(f+g)T(\psi)$$

$$= T(\varphi f\psi) + T(\varphi g\psi)$$

$$\quad - T(\varphi)T(f)T(\psi)$$

$$\quad - T(\varphi)T(g)T(\psi)$$

$$= T(\varphi f)T(\psi) + H(\varphi f)H(\tilde{\psi})$$

$$+ T(\varphi)T(g\psi) + H(\varphi)H(g\tilde{\psi})$$

$$- T(\varphi f)T(\psi) + H(\varphi)H(\tilde{f})T(\psi)$$

$$- T(\varphi)T(g\psi) + T(\varphi)H(g)H(\tilde{\psi})$$

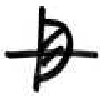
= a trace class operator

We also know that

$$T_n(\varphi)^{-1} \longrightarrow T(\varphi)^{-1}$$

strongly when $|\mathrm{Re}\,\beta_2| < \frac{1}{2}$

(If $\beta = \frac{1}{2}$ $T(\varphi)$ is not invertible



)

Also $(T_n(\varphi)^{-1})^* \longrightarrow (T(\varphi))^*$

so from our properties

$$K_n \quad P_n H(\varphi) H(\tilde{\varphi}) P_n T_n(\tilde{\varphi})^{-1} T_n(\varphi)^{-1}$$

$$\longrightarrow H(\varphi) H(\tilde{\varphi}) T(\tilde{\varphi})^{-1} T(\varphi)^{-1}$$

in the trace norm

What about

$$J_n = B_n T_n(\psi)' T_n(\varphi)^{-1} ?$$

$$B_n = W_n H(\tilde{\varphi}) H(\psi) W_n$$

$B_n \to 0$ but only in the "strong" sense

So we have

$$I_n + K_n + J_n$$

$$= I_n + K_n + J_n + K_n J_n - K_n J_n$$

$$= (I_n + K_n)(I_n + J_n) - K_n J_n$$

So $\det(I_n + K_n + J_n)$

$\sim \det(I_n + K_n)(I_n + J_n)$

since $K_n J_n \longrightarrow$

in the trace norm

and the inverses of

$(I_n + K_n)(I_n + J_n)$ are

uniformly bounded

Thus we need

$$\lim_{n \to \infty} \det(I_n + K_n)$$

and

$$\lim_{n \to \infty} \det(I_n + J_n)$$

$$\det(I_n + K_n)$$

$$= \det(I_n + P_n H(\varphi) H(\tilde{\varphi}) P_n T_n(\psi)^{-1} T_n(\psi)^{-1})$$

$$\longrightarrow \det(I + H(\varphi) H(\tilde{\varphi}) T(\psi)^{-1} T(\psi)^{-1})$$

$$= \det(I + (T(\varphi\psi) \quad T(\varphi) T(\psi)) T(\psi)^{-1} T(\varphi)^{-1})$$

$$- \det T(\varphi\psi) T(\psi) \quad T(\psi)^{-1}$$

$$\det(I_n + J_n)$$

$$= \det(I_n + W_n H(\tilde{\varphi}) H(\psi) W_n T_n(\psi)^{-1} T_n(\varphi)^{-1})$$

$$\det(I_n + P_n H(\tilde{\varphi}) H(\psi) W_n T_n(\psi)^{-1} W_n^{-2} T_n(\psi)^{-1} W_n)$$

$$\det(I_n + P_n H(\tilde{\varphi}) H(\psi) T_n(\tilde{\psi}) T_n(\tilde{\varphi})^{-1})$$

This last term converges to

$$\det\left(T(\tilde{\varphi}\tilde{\psi})\,T(\tilde{\psi})^{-1}T(\tilde{\varphi})^{-1}\right)$$

To summarize

$$D_n(\varphi\psi)\,/\,D_n(\varphi)D_n(\psi)$$

$$\longrightarrow$$

$$\det\left(T(\varphi\psi)\,T(\psi)^{-1}\,T(\varphi)^{-1}\right)$$

$$\times \det\left(T(\tilde{\varphi}\tilde{\psi})\,T(\tilde{\psi})^{-1}T(\tilde{\varphi})^{-1}\right),$$

as $n\to\infty$

Note This same proof works to Block matrices and finite Wiener-Hopf operators

Let's go back to the symbol

$$\psi(e^{i\theta}) = \varphi(e^{i\theta}) \prod_{j=1}^{R} \varphi_{\alpha_j, \beta_j}(e^{i(\theta-\theta_j)})$$

$D_n(\varphi)$ follows from the Szegö Limit Th

$$D_n(\varphi_{\alpha,\beta}) = \frac{G(1+\alpha+\beta)\, G(1+\alpha-\beta)\, G(1+n)\, G(1+2\alpha+n)}{G(1+2\alpha)\, G(1+\alpha+\beta+n)\, G(1+\alpha-\beta+n)}$$

$$\sim n^{\alpha^2 - \beta^2} \frac{G(1+\alpha+\beta)\, G(1+\alpha-\beta)}{G(1+2\alpha)}$$

From this the conjecture follows

To do the case where $|R\alpha_2| < \frac{1}{2}$ one must consider different "spaces" (not H^2) but the ideas are the same

There are other regions where the conjecture is true, say for $R\alpha_2 > -\frac{1}{2}$ & all $\beta_2 = 0$

But it turns out that the conjecture is not always true

Consider this example

$$\psi(e^{i\theta}) = \begin{cases} -1 & -\pi < \theta < 0 \\ +1 & 0 < \theta < \pi \end{cases}$$

$D_n(\psi) = 0$ if n is odd

$$D_n(\psi) \sim (i)^n \, n^{\frac{1}{2}} \, 2^{\frac{1}{2}} \, G(\tfrac{1}{2})^2 \, G(\tfrac{3}{2})^2$$

as $n \to \infty$ for n even

Notice

$$\psi(e^{i\theta}) = \varphi_{0,\frac{1}{2},0} \; \varphi_{0,\frac{1}{2},\pi}$$

$$= \varphi_{0,-\frac{1}{2},0} \; \varphi_{0,\frac{1}{2},\pi}$$

The conjecture does not even make sense since there is more than one way to describe φ using the standard factors

In general, if

$$\psi = \varphi \cdot \prod_{j=}^{R} \varphi_{\alpha_j \beta_j, \theta_j}$$

then $\psi = \varphi^* \prod_{j=}^{R} \varphi_{\alpha_j, \beta_j + n_j, \theta_j}$

where $\sum n_j = 0$

$$\varphi^* = \varphi \prod_{j=1}^{R} \left(e^{i \theta_j} \right)^{n_j}$$

In the example,

$\beta \quad \frac{1}{2}$, $\quad \beta_2 \quad \frac{1}{2}$

$\theta = 0 \qquad \theta_2 = \pi$

$n \quad 1 \qquad n_2 = 1$

So here is a modified conjecture

Suppose

$$\psi(e^{i\theta}) \quad g^k \prod_{j=}^{R} \varphi_{\alpha_j^k, \beta_j^k, \theta_j}$$

Define $Q(k) \quad \sum_{j=1}^{R} (\alpha_j^k)^2 - (\beta_j^k)^2$

Let $Q \quad \max \operatorname{Re} Q(k)$

$$\mathcal{K} = \{ k \quad \operatorname{Re} Q(k) \quad Q \}$$

Then the generalized conjecture says that

$$D_n(\psi) \quad \sum_{k \in \mathcal{K}} G(\varphi^k)\, n^{Q(k)} E_k$$

$$+\ o\left(|G(\varphi)|^n n^{Q} \right)$$

($\alpha_j + \beta_j$, $\alpha_j - \beta_j$ should not be negative integers)