Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Spacing distributions for random matrix ensembles II

P. Forrester (Melbourne)

06 April 2004

Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK

Tel: +44 1223 335999 Fax: +44 1223 330508
E-mail: webseminars@newton.cam.ac.uk
http://www.newton.cam.ac.uk/webseminars

Okamoto τ-$f^{\underline{n}}$ theory of Painlevé systems

- **Motivation**

N eigenvalues on the real line

Joint PDF $\quad p(x_1, x_2, \dots, x_N)$

Seek formulas for

(A) $\Pr\big($ $[a_1, a_2]$ contains exactly n eigenvalues$\big)$

(B) $\Pr\big($exactly n eigenvalues to the right of x $\big|$ there is an eigenvalue at $x\big)$

Have

$$(A) = \binom{N}{n} \int_J dx_1 \cdots \int_J dx_n \int_{\bar{J}} dx_{n+1} \cdots \int_{\bar{J}} dx_N \, p(x_1, \dots, x_N)$$

$[a_1, a_2]$ $\qquad$ $(-\infty,\infty)\setminus J$

$$= \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \xi^n} \underbrace{E_N(J;\xi)} \Big|_{\xi=1}$$

$$\left(\int_{-\infty}^{\infty} - \xi \int_J \right) dx_1 \cdots \left(\int_{-\infty}^{\infty} - \xi \int_J \right) dx_N \, p(x_1, \dots, x_N)$$

Generating $f^{\underline{n}}$

$$(B) - \ {}^{N}_{n} (N-n-1) \int_{J^{n} \sim (x,\infty)} dx \quad \int_{J} d \ {}^{n+1} \int_{J} dx \ {}_{2} \int_{J} dx \ p(x\ x_2, \ldots x_N)$$

$$(1)^n \frac{\partial}{\partial \xi} \underbrace{p_N \quad \xi)}_{} \quad \xi = 1$$

$$\left(\int_{\infty}^{\infty} - \xi \int_{J}\right) dx_2 \quad \left(\quad \xi_{J}\right) dx_N \ p(\quad x_2 \quad ., x_N)$$

An ther generating $f^{\underline{n}}$

S pp se w

$$p(x_1 \quad x_N) \propto \prod_{j=1}^{N} g(x \ ) \prod_{1 \le j \ k \ N} (x_k \ x \ )^2$$

(Ca l th s $UE_N \ g)$

Define

$$E_N \ \mu \ J \ \xi \ g) \quad \left\langle \prod_{l=1} 1 \ \xi \chi_J^{(l)} \quad x \ x_l \ {}^{\mu} \right\rangle_{UE_N \ g}$$

$$\frac{1}{C}\left(\int \quad \xi \int_{J}\right) dx_1 \quad \left({}^{\infty} \quad \xi \int_{J}\right) dx_N$$

$$\prod_{1}^{N} g(x \ ) \quad x \ x \ {}^{\mu} \prod_{1 \le j < k \le N} (x_k - x_j \ {}^{2}$$

The

$$(A) - \ E_N \quad 0 \ J \ \xi \ g$$

$$(B) \propto \ g(c) \ E_N \quad 2 \ J \ \xi \ g)$$

Also, for $\xi = 0$ have

$$\left\langle \prod_{l=1}^{N} |x - x_l|^{\mu} \right\rangle_{UE_N(g)} = \text{average of power of characteristic polynomial}$$

- Can characterize such averages a τ-f$^{\underline{n}}$ for a Painlevé system in the cases that g is classical:

$$\frac{d}{dx} \log g(x) = \frac{p(x)}{q(x)}$$

$p(x)$: poly. degree ≤ 1

$q(x)$: poly. degree ≤ 2

$$\Rightarrow g(x) = \begin{cases} e^{-x^2} \\ x^a e^{-x}, & x > 0 \\ (1-x)^a (1+x)^b, & -1 < x < 1 \\ (1+x^2)^{-\alpha} \end{cases}$$

- <u>Hamiltonian formulation of the Painlevé theory</u>

A. Relationship to Painlevé transcendents

Hamiltonian $\quad H = H(p, q, t, \vec{v})$

q: Painlevé transcendent

$\vec{v}$: parameters

Hamilton eq$^{\underline{ns}}$ ① $q' = \dfrac{\partial H}{\partial p}$ ② $p' = -\dfrac{\partial H}{\partial q}$

Eliminating p gives Painlevé d.e. in q

e.g. $H_{II} = -\frac{1}{2}(2q^2 - p + t)p \quad \frac{v_1 - v_2}{2} q$

① $q' = -\frac{1}{2}(2q^2 - 2p + t)$ ② $p' = -2qp - \frac{v_1 - v_2}{2}$

① $\Rightarrow \quad -2q' = 2q^2 - 2p + t$

$\Rightarrow \quad p = q' + q^2 + t/2$

diff. ①

$$a'' = -2qq' - p' - \frac{1}{2}$$

$$= -2qq' + 2pq + \frac{v_1 - v_2}{2} - \frac{1}{2}$$

$$= -2qq' + 2q\left(q' + q^2 + \frac{t}{2}\right) + \frac{v_1 - v_2}{2} - \frac{1}{2}$$

$$= 2q^3 + qt + \frac{v_1 - v_2 - 1}{2}$$

Painlevé II DE with $\alpha = \frac{v_1 - v_2 - 1}{2}$

$$q'' = 2q^3 + qt + \alpha$$

B. DE satisfied by the (auxilary) Hamiltonian

Hamilton eq$^{n\,3}$ $\Rightarrow \quad \frac{dH}{dt} = \frac{\partial H}{\partial t}$

e.g. For PII

$$\frac{dH_{II}}{dt} = -\frac{1}{2}p \qquad (a)$$

using ①

$$\Rightarrow \quad \frac{d^2 H_{II}}{dt^2} = -\frac{1}{2}p' = \frac{1}{2}\frac{\partial H_{II}}{\partial q} = \frac{1}{2}\left(-2qp - \frac{v_1 - v_2}{2}\right) \qquad (b)$$

Also

$$t\frac{dH_{II}}{dt} \quad \frac{1}{2} \; 2q^2 \; p)p \quad \frac{v_1 \; v_2}{2} q \qquad c)$$

$$(b) \Rightarrow \left(\frac{d^2 H_{II}}{dt^2}\right) - qp + \frac{v_1 \; v_2}{2} pq + \frac{(v_1 - v_2)}{4^2}$$

using (c) $\longrightarrow$

$$p\left(H_{II} \quad t\frac{dH_{II}}{dt} \quad \frac{1}{2}p\right) + \frac{(v_1 \; v_2)^2}{4^2}$$

$$-2\frac{dH_{II}}{dt}\left(H_{II} \quad t\frac{dH_{II}}{dt} \quad 2\left(\frac{dH_{II}}{dt}\right)^2\right) + \frac{(v \; v_2)^2}{4^2}$$

Set $\quad \sigma_{II} \; t) \quad 2^{1/3} H_{II}(2^{1/3} t) \qquad (v \; v_2 \quad (a \quad a$

$$\Rightarrow \underbrace{(\sigma_{II})^2 + 4\,\sigma_{II} \quad (\sigma_{II})^2 \quad t\sigma_{II} + \sigma_{II}) - a^2 \quad 0}_{\text{form f P inleve II}}$$

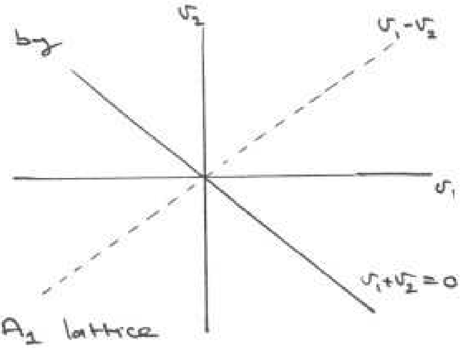
Note C express H_{II} d therefore σ_{II}
in terms f q q

Painleve II transcendent with α a $1/2$

C. Aff. Wey group symmetries

σPII s ch ged by

$s_1(v_1 \; v \quad (v \quad v_1)$

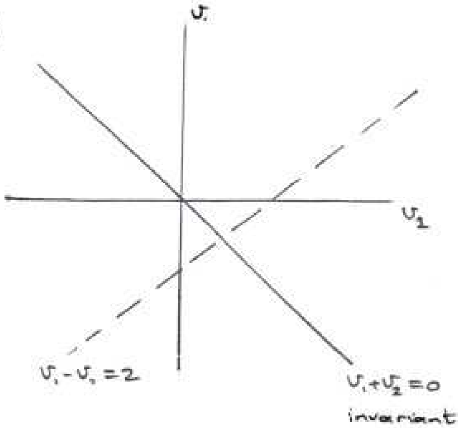


reflection the
line $v \quad v$

$v + v_2 \quad 0 \quad n \quad t$
Direction f $\hat{e}_1 \; \hat{e}_2$ root f A_1 lattice

There also ther f. damenta reflection mapping.

$s_0(v_1 \ v \quad 2+v. \quad 2+v_1)$



A gebra (ffine Wey group A_1)

$s_0^2 \quad 1 \qquad s_1 \quad 1 \qquad s_0 s_1)^4 - 1$

Suggests another fundamenta pping

rot. tion f 180 about $v_1 \ v_2) - 1, 1$
(⇔ r. flectio that pt

$\pi \ v_1 \ v_2 = 2 \ v_1 \quad 2 + v_2)$

Thi nterchange r. flection lines leaves parameter space $v_1 + v_2 = 0$ inva t

Algebra

$$\pi^2 = 1, \qquad \pi s_0 = s_1 \pi$$

$\langle \pi, s_0, s_1 \rangle$ Extended A_1 Weyl algebra

D. Bäcklund transformations

Action of π, s_0, s_1 is defined on the parameters

- Can define action of π, s_0, s_1 on p, q such that the Hamilton eq^{ns} are conserved

e.g. PII Define scalars

$$\underbrace{\alpha_0 = 1 + \frac{v_2 - v_1}{2}}_{\substack{\alpha_0 = 0 \text{ defines} \\ \text{reflection line} \\ \text{for } s_0}}, \qquad \underbrace{\alpha_1 = \frac{v_1 - v_2}{2}}_{\substack{\alpha_1 = 0 \text{ defines} \\ \text{reflection line} \\ \text{for } s_1}}$$

Have constraint $\alpha_0 + \alpha_1 = 1$

	α_0	α_1	p	q
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$p + \frac{4\alpha_0 q}{f} + \frac{2\alpha_0^2}{f^2}$	$q + \frac{\alpha_0}{f}$
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	p	$q + \frac{\alpha_1}{p}$
π	α_1	α_0	$-f$	$-q$

$p - 2q^2 - t$

Derived by Okamoto, Noumi & Yamada

Note Must have

$s_i^2 p - p \qquad \pi^2 p \quad p \qquad$ etc

e.g. $\pi^2 p - \pi(f)$

$\pi p \quad 2(\pi q)^2 \quad t$

$(f \quad 2q^2 \quad t) \qquad p$

Can check that

$s_0 H_{II} \qquad H_{II} + \frac{\alpha_0}{f}$

$s_1 H_{II} \qquad H_{II}$

$\pi H_{II} \qquad H_{II} + q$

E/ Shift (Schle. nger) oper to s

These are composition of the elementary operators which shift the s by 1 0

e.g $T_{II} - \pi s_1$

$T_{I} \; \alpha. \; \alpha) = \pi s_1 \; \alpha_0 \; \alpha_1)$

$(\alpha \quad 2\alpha_1, \quad \alpha)$

$- (\alpha + 2\alpha \quad \alpha_0) \qquad$ but $\alpha + \alpha_1 \quad 1$

$(1 \quad \alpha_0 \quad 1 + \alpha)$

$\Leftrightarrow \; T_{I} \; v_1 \; v) - (v \quad 1 \quad v_2 + 1)$

- It turns out that the shift operators of interest also have the property

$$T H\Big|_{\substack{\vec{J}=\vec{J}^{(n)}\\ p=p[n]\\ q=q[n]}} := H_{\substack{\vec{J}=\vec{J}^{(n+1)}\\ p=p[n+1]\\ q=q[n+1]}} = H_{\substack{\vec{J}=\vec{J}^{(n+1)}\\ p=p[n]\\ q=q[n]}}$$

write

$$T^n H_{\substack{\vec{J}=\vec{J}^{(0)}\\ p=p[0]\\ q=q[0]}} =: H[n]$$

Note that $H[n]$ satisfies the appropriate σ-form of a Painlevé eq^{n}.

- Introduce the τ-f^{n} by

$$H[n] = \frac{d}{dt} \log \tau[n]$$

<u>Essential point</u> Can inductively construct $\tau[n]$ obtaining

$$\Big\langle \prod_{l=1}^{n} (1-\chi_J^{(l)}) |x-x_l|^{\mu} \Big\rangle_{\text{U}E_n(g)}$$

g classical, amongst other examples. Hence can characterize such averages as the sol^{n} of a σ-form Painlevé eq^{n}

F Cla ical solutions

Wa t $\tau[\ ]$ 1 d thn $H[\ ]$ 0

n. o random matrix average

By restri t g the parameter spa to particular chamber wa the Painle e sy tems pe t such

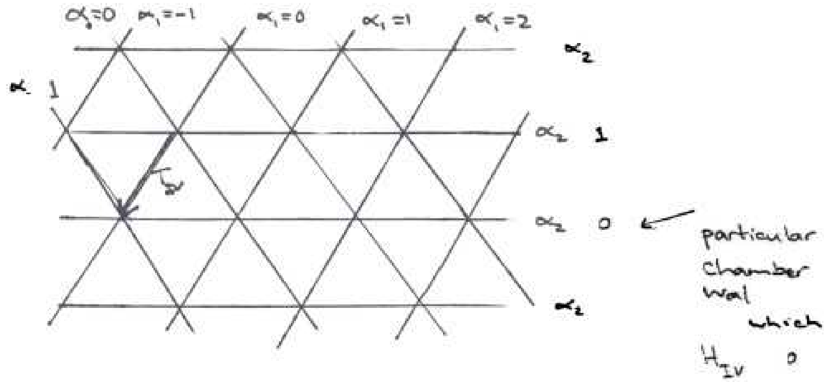
e.g. PIV r. t yst parameters

α_0 = 1 + v_3 v_1

v_1 v_2

α_2 v_2 v_3

$\underbrace{\alpha + \alpha_1 + \alpha_2 \quad 1}_{\text{2-d space}}$



T_{IV} $\alpha.$ α $\alpha.$ α 1 α α_2 1

T_{IV} v $v.$ v $)$ $=$ v $\frac{1}{3},$ $v.$ $\frac{}{3}$ $v_3 + \frac{2}{3}$

H_{IV} $(2p$ q $2t$ pq 2 v $v_2)p + (v_3$ $v_2)q$

Suppose $\alpha_2 = 0$ i.e v_2 0

Then we see that setting $p-$ gives

$H_{IV} - 0$

Also

$H_{IV}[1]$ T_{IV} $H_{IV}[0]$ H_{IV} $\vec{v}$ $\vec{v}^{(1)}$ p $p[0]$ $q = q[\ ]$

H_{IV} $\vec{v}$ $\vec{v}^{(}$ $+ q[\ ]$

$q[\ ]$

But

$q[\ ]$ $\frac{\partial H}{\partial p}\Big|_{\substack{p= \\ \vec{v}\ \ v^{(0}}}$

q $4p$ q $2t$ 2 $v^{(0}$ $v^{(0)}\Big|_{q\ \ q[0}$

$-$ $q[0]$ $q[0] + 2t)$ 2 v $v_2^{(0)}$

R tt. eq

Line ised by $q[.] - \frac{d}{dt}$ og $\tau[1]$

$\Rightarrow$ $\tau[1]$ satsfies

$u'' + 2tu + 2(v^{(0}$ $v_2^{(0)})u - 0$

Hermite Weber eq^n

Th per to the genera integra sol^n

$$\left(\infty \quad \xi \int_t^\infty \right) t \quad u \quad v^{(0)} v_2^{\,\cdot 1} \, e \quad u^2 \, du$$

$$\left\langle \; t - u \quad {}^{\mu} \right\rangle_{UE_1 \, e \, u^2)}$$

set

$v^{(0)} \; v^{(0)}) \quad \mu$

Q. Toda lattice eq^n

what about $\tau[\;]$ for $n \geq 2$?

establi h the Toda lattice eq

$$\frac{d^2}{dt^2} \; g\left(e^{\,t^2(\alpha_2^{(0)} \; n} \; \tau_{IV}[n]\right) - \frac{\tau_{IV}[n+1] \, \tau_{IV}[\quad 1]}{(\tau_{IV}[n])^2}$$

Method

$$\frac{d}{dt} \log \frac{\tau_{IV}[n+1] \, \tau_{IV}[n-1]}{(\tau_{IV}[n])^2} - H_{IV}[n+\;] + H_{IV}[\quad 1] \quad 2 H_{IV}[n]$$

$$(T_{IV} H_{IV}[n] \quad H_{IV}[n]) + T_{IV}^{\,1} (H_{IV}[n] \quad T_{IV} H_{IV}[n]$$

$$- q[n] \quad \underbrace{T_{IV}^{\,1} q[n]}$$

compute th from action f the elementary operators using Table

verify from explicit calc $\longrightarrow$

$$\frac{d}{dt} \log \left(\frac{d}{dt} H_{IV}[\;] \quad t(\alpha_2^{(0)} \; n) \right)$$

$$\frac{d}{dt} \log \frac{d}{dt^2} \log\left(\tau_{IV}[n] \, e^{\,t^2(\alpha_2^{(0)} - n)}\right)$$

S g ficance

result Sylvester derived using properties inor f deter inants gives that the Toda lattice eq$^{\underline{n}}$

$$\frac{d}{dt^2} \cdot g\, g[n] \quad \frac{g[n+1]\, g[n-1]}{g^2[n]}$$

ha the ique so$^{\underline{n}}$

$$g[n] \quad \underbrace{\det\left[\frac{d^{j+k}}{dt^{j+k}} g[1]\right]_{j,k}}_{\text{Double Wron kian}} \qquad 1$$

$$\tau_{IV}[n] \quad \det\left[\frac{d^{j+k}}{dt^{j+k}} e^{t^2} \tau_{IV}[1]\right]_{j,k} \qquad 1$$

$$\left(\int_{-\infty}^{\infty} \; 3\int_t^{\infty}\right) t \;\; u)^{-(v \quad v_2^{(\infty)}} e^{\;u^2} du$$

Can eas ly perform the d fferentiations

$$e^{t^2}\left(\int^{\infty} \; 3\int_t^{\infty}\right)(t-u^{\;(v \quad v^{(\infty)}} e \;\; du$$

$$(\quad 3\int^{\infty})\,(u \quad v^{(\infty)} \quad {}^{(\infty)})\, e \;\; u \;\; 2tu \; d$$

$$\frac{d^{j+k}}{dt^{j+k}} \quad) \quad \left(\int^{\infty} \quad \int^{\infty}\right)(u)^{(v \quad v_2} e^{\;u^2}(2u)^{j+k} e^{2tu}$$

$$2^{j+k} e^{t^2}\left(\int^{\infty} \; 3\int_t^{\infty} \quad u\right)^{(v^{(\infty)} v_2 \;+\; k} e \;\; du$$

$$\rightarrow \tau_{IV}[n] \quad 2^{n(n-} \det\left[\int^{\infty} \; 3\int_t^{\infty} (t \quad)^{v_1^{(\infty)} v_2 \;)+j+k} e^{\;u} du\right]$$

in genera

$$\det\left[\int_{\infty}^{\infty} f \quad t\ u)^{j}\ {}^{k} du\right]_{j,k=0 \quad n\ 1}$$

$$= \int^{\infty} du\ f(u) \quad {}^{\infty} du_n f(u_n)\ \det\left[\ t\ u_{\cdot})^{j+k}\right]_{j,k \quad ,n n}$$

$$= \int^{\infty} du\ f(u) \quad {}^{\infty} du_n f \quad n) \prod \quad t\ u_{j+1})^{j-1}$$

$$\underbrace{\det\left[\ t\ u_{j+1})^{k}\right]_{j,k \quad n}}$$

$$\prod_{1 \quad j<k \leq n} \Big(\quad \underbrace{u_k \qquad u}_{u_j \quad u_k}$$

Vandermonde for la

$$\frac{1}{n} \int^{\infty} du_1 f(u \ ) \quad \int^{\infty} du_n f \quad n) \prod_{\leq j<k \leq n} \quad u - u_k)^2$$

symmetrise integrand use Vandermonde

$$\tau_{IV}[n] = \frac{2^{n(n-1)}}{n} \left(\int_{-\infty}^{\infty} \ \xi \int_{t}^{\infty}\right) du_1 e^{\ u} \quad t.\ u)^{-(v^{(0)}\ v_2^{(0)})}$$

$$\left(\int^{\infty} \ \xi \int_{t}^{\infty}\right) du_n e^{\ u_n} (t \qquad)^{\ (v^{(0)}\ v}$$

$$\prod_{1 \leq j<k \leq n} \quad u_k \qquad)$$

$$\propto \left\langle \prod \left(1 \quad \xi\, \chi^{(y)}_{(t,\infty)} \quad t\ x\ \right)^{-(v \quad v^{(0)})} \right\rangle_{UE_n(e^{-x^2}}$$

As before   case) choose $v^{(0)}$ $v_2^{(0)}$ μ

Consequence

$$U_N^G \; t \; \mu \; \xi \; = \frac{d}{dt} \; og \left\langle \prod_{j=1}^{N} \; 1 \; \xi \chi^{(j)}_{(t,\infty)} \right) \; t \; x_j \; ^{\mu} \Big\rangle_{UE_N(e^{-x^2})}$$

satisfies the σPIV eq

$$(\sigma_{IV}'')^2 \; 4 \; t\sigma_{IV}' \; \sigma_{IV} \;)^2 + 4\sigma_{IV}'(\sigma_{IV}' + 2\alpha)(\sigma_{IV}' \; 2\alpha_2) - 0$$

th

α μ α N

Boundary condition known

$\xi = 0$ μ

ξ 1 μ 2

e g μ 0

$$U_N^G \; t \; 0 \; \xi) \underset{t\to\infty}{\sim} \xi \rho_N^G$$

$e \; x^2$

$$\langle p \;\; p_{N-1} \rangle \; (p \;\; (x) p_{N-} (x) \;\; p_{N-} (x) p_N(x))$$

$2 \; ^N H_N(x$

$$\Big\langle \prod_{j=1} (1 \; \xi \chi^{(j)}_{(t,\infty)} \Big\rangle_{UE_N(e^{-x^2}} \; - \exp\Big(\int_t U_N^G (y, \;\; \xi) dy \Big)$$

H

Application to the gener tion f smal t

expa ions J cobi ca

$\prod_{l=}^{N}(1 \quad \xi \chi^{j} \quad t)) \Big>_{U\in_N} x^a(\quad x)^b$

$$t \{ \underbrace{U_N^J \quad y; a \; b \; o \; \xi)}_{} \quad c\,t + c_2 \quad \frac{dt}{t(t-1)}$$

k_1 expl Hy

f genera μ $\langle$ $t \; x_1 \; \mu$ $\rangle$ atisfies

σPVI q^{n}

$$\sigma_{VI}\left(t(\quad t \; \sigma_{VI}\right)^2 + \left(\sigma_{VI}\{2\sigma_{VI} \; (2 \quad _{VI} \quad + v_1 v \, v_3 v_4\right)^2$$

$$= \prod_{k}^{4}\left(\sigma_{VI} + v_k^2\right)$$

th

$v \quad \frac{}{2}\; a \cdot b \quad N+\mu \qquad v_3 = \frac{1}{2}(a \quad b \quad + N$

$v_2 = \frac{}{2}(a+b) \qquad v_4 \quad \frac{}{2} \; b \quad a)$

Bo dary ndition

$$f\,t)_{t\to\,+} \; -\xi\, c\, t^{a+1}$$

kn e. Hy

The DE generate the small t expans

$$f(t)_{t\to +} \quad \xi c t \quad \left(1 + c^{1}\, t + c \quad t^2 + \qquad \right)$$

$$\frac{(\xi c)^2\, t^{2a+2}}{1+a}\left(1 + c_2^{(1)} t + c_2 \quad t^2 + \qquad\right)$$

$$\frac{(\xi c)^3\, t^{3a+3}}{(1+a)^2}\left(1 + c_3^{1}\, t + c^{2} \quad t^2 + \qquad\right)$$

$$\Rightarrow \left\langle \prod_{l=1}^{N} \left(1 - \xi \chi^{(l)}_{(0,t)}\right) \right\rangle_{UE_N(x^a(1-x)^b)}$$

$$= 1 - \frac{c}{1+a} \xi t^{a+1} \left(1 + b_1^{(1)} t + b_1^{(2)} t^2 + \cdots\right)$$

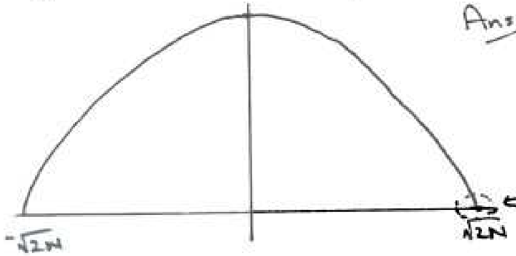
$$+ A_1 \xi^2 t^{2a+4} \left(1 + b_2^{(2)} t + b_2^{(2)} t^2 + \cdots\right)$$

$$+$$

general "leading" term t^{ka+k^2}

I. Soft edge scaling

Reconsider the Gaussian case – what is the eigenvalue density?



Ans. Wigner semi-circle law

$$\rho(x) \underset{N\to\infty}{\sim} \frac{\sqrt{2N}}{\pi} \sqrt{1 - \frac{x^2}{2N}}$$

call this the soft edge

should scale about this pt

$$x \longmapsto \sqrt{2N} + \frac{X}{\sqrt{2}\, N^{1/6}}$$

Same scaling as Hermite to Airy transition

$\therefore$ Consider

$$E^{\text{soft}}(s, \xi) = \lim_{N\to\infty} \left\langle \prod_{j=1}^{N} 1 - \xi \chi^{(j)}_{(t,\infty)} \right\rangle_{UE_N(e^{-x^2})} \Bigg|_{t \mapsto \sqrt{2N} + \frac{s}{\sqrt{2}N^{1/6}}}$$

$$= \lim_{N\to\infty} \exp\left(- \frac{1}{\sqrt{2}N^{1/6}} \int_s^\infty U_N^G\left(\sqrt{2N} + \frac{t}{\sqrt{2}N^{1/6}}; 0; \xi\right) dt \right)$$

Suggests the existence of the scaled transcendent

$$u^{soft}(t;0;\xi) = \lim_{N\to\infty} \frac{1}{\sqrt{2}N^{1/6}} \underbrace{U_N^G\left(\sqrt{2N} + \frac{t}{\sqrt{2}N^{1/6}}; 0; \xi\right)}_{\substack{\text{satisfies } P\sigma_{IV} \text{ eq}^{n} \text{ with} \\ \alpha_1 = 0,\ \alpha_2 = -N}}$$

Then would have

$$E^{soft}(s;\xi) = \exp\left(- \int_s^\infty u^{soft}(t;0;\xi)\, dt \right)$$

Formally taking the limit in the d.e. gives that u^{soft} satisfies

$$(\sigma_{II}'')^2 + 4\sigma_{II}'\left((\sigma_{II}')^2 - t\sigma_{II}' + \sigma_{II} \right) - \frac{1}{4}\Big(\underset{\substack{\uparrow \\ \text{parameter} \\ \text{in } PII}}{\alpha} + \frac{1}{2}\Big)^2 = 0$$

with $\alpha = -\frac{1}{2}$. Must be solved subject to the boundary condition

$$u^{soft}(t;0;\xi) \underset{t\to\infty}{\sim} \xi \underbrace{\left((Ai'(s))^2 - s(Ai(s))^2 \right)}_{\substack{\text{scaled limit of} \\ \text{corresponding boundary} \\ \text{condition for } U_N^G.}}$$

- Expression for $\sigma_{\rm II}$ in terms of Painlevé II transcendent

$$\sigma_{\rm II}(t) = -2^{1/3} H_{\rm II}(-2^{1/3}t)$$

where $$H_{\rm II} = -\tfrac{1}{2}(2q^2 - p + t)p - (\alpha + \tfrac{1}{2})q$$

and q satisfies PII :

$$q'' = tq + 2q^3 + \alpha$$

while p is related to q by

$$q' = \frac{\partial H}{\partial p} = p - \frac{t}{2} - q^2 \quad \Rightarrow \quad p = q' + q^2 + \frac{t}{2}$$

It follows that

$$H_{\rm II}' = -\tfrac{1}{2}p = -\tfrac{1}{2}(q' + q^2 + \tfrac{t}{2})$$

and so

$$u^{\rm soft}(t;0;\xi) = -\frac{1}{2^{1/3}}\Big[q'(t;-\tfrac{1}{2}) + q^2(t;-\tfrac{1}{2} + \tfrac{t}{2}\Big]_{t \mapsto -2^{1/3}t}$$

But an identity due originally to Gambier gives

$$-\varepsilon 2^{1/3} q^2(-2^{-1/3}t;0) = \frac{d}{dt}q(t;\tfrac{\varepsilon}{2}) - \varepsilon q^2(t;\tfrac{\varepsilon}{2}) - \frac{\varepsilon t}{2}, \quad \varepsilon = \pm 1$$

and so $$\frac{d}{dt}u^{\rm soft}(t;0;\xi) = -q^2(t;0)$$

Hence $\quad F_t(s\ 3) \quad ex \quad \int_s^\infty t \cdot s\ q(t) dt)$

where q satisfies

$$q \quad tq + 2q^3 \qquad (P\mathrm{II}\ eq^n \text{ with } \alpha\ 0)$$

bject t. the soundary cond'n

$$q \quad 3\,Ai(t) \quad \text{as } t \to \infty$$

(Hastings Macleod sol)