Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Computational methods for L-functions II

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Euer Maclarn smmatin works

well for $\zeta(s)$ and alo for Dirhlet

L fu tions $L(s,\chi)$ because the Dirichlet

coefficients are very smpe

Not very efficient For $\zeta(s)$ # terms needed

$O(|s|)$

Much better to use Reman Sege

or smoothed variants

Riemann Siegel Formula

$$Z(t) = e^{i\theta(t)}\zeta(\tfrac{1}{2}+it)$$

$$e^{i\theta(t)} = \left(\frac{\Gamma(\frac{1}{4}+\frac{it}{2})}{\Gamma(\frac{1}{4}-\frac{it}{2})}\right)^{\frac{1}{2}}\pi^{-\frac{it}{2}}$$

$Z(t)$ is real

Riemann-Siegel

$$Z(t) = \underbrace{2\sum_{1}^{N} n^{-1/2}\left(\cos(\theta(t)-t\log n)\right)}_{\text{main sum}} + R(t)$$

$$R(t) \sim \frac{(-1)^{N+1}}{a^{1/2}}\sum_{0}^{\infty}\frac{c_r(p)}{a^r}$$ asymptotic series (Berry studied asymp. of the c_r's)

$a = (t/2\pi)^{1/2}$ $\quad N = \lfloor a \rfloor$ $\quad p = \{a\}$

(Formula can be extended to $s = \sigma + it$ see Odlyzko-Schonhage for summary references)

$$C_1(\rho) = \Psi(\rho)\ \cos(2\pi(\rho^2 - \rho - 1/16))/\cos(2\pi\rho)$$

$$C(\rho) = \frac{}{9\pi^2}\Psi^{(3)}(\rho)$$

$$C_2(\rho) = \frac{1}{1842\pi^4}\Psi^{(6)}(\rho) + \frac{1}{64\pi^2}\Psi^{(2)}(\rho)$$

$$C_3(\rho) = \frac{-1}{53084\pi}\Psi^{(9)}(\rho)\ \frac{1}{3840\pi^4}\Psi^{(5)}(\rho)\ \frac{1}{64\pi^2}\Psi$$

R emann used sadd poi t method to obta n C_j $j \le 5$ Reason such n ce formulae are poss ble using harp cut off s that all the D richlet coefficients are R emann starts with a representat n for $\zeta(s)$ that nv ves geometric series dentity $1\ x = \sum x^n)$

Dir hlet coefficients of ζ

F r a genera L f nction sm othing is better

Titchmarsh showed error is $< \frac{3}{2}\left(\frac{t}{2\pi}\right)^{3/4}$

if $\frac{t}{2\pi} > 25$, and we take just the

C_0 term

Gabcke showed error is $< .053\, t^{5/4}$

if $t > 200$ and we take just the C_0 and

C_1 terms

Odlyzko & Schönhage developed an

algorithm to compute the main sum

for $T \leq t < T + T^{1/2}$ in $O(t^{\epsilon})$ operations

provided a precomputation involving $O(T^{1/2+\epsilon})$

operations and bits of storage are carried out

beforehand. This algorithm enabled

Odlyzko's monumental computations

Main Lesson

Sm th things are nice

Smoothing as a techinque n analytic nu ber theory is over 00 years old

de la Vallee Poussin's proof of the prime number theorem volved showing

$$\psi(x) \quad \sum_{n<x} (x-n)\Lambda(n) \quad \int^{x} \psi(t \, dt$$

s asymptotic to $x^2/2$

$$\left(\psi \ x) \sim \frac{x^2}{2}\right) \Leftrightarrow \psi(x) \sim x\Big)$$

$$2\pi i \int_{(c)} \frac{y^s \, ds}{s \ s \ )} \quad \begin{cases} y & y \geq \\ 0 & y \leq \end{cases} \qquad c > 0$$

$$\frac{\psi(x)}{x} - \sum_{n<x} \Big(\ \frac{n}{x}\Big)\Lambda \ n) \quad \frac{1}{2\pi i} \int_{(c} \frac{x^s}{s(s+1} \left(\frac{\zeta'(s)}{\zeta(s} \right)$$

extra factor
gives convergent integra
on the 1 line (away fr

70's–2000's Many authors have used Lavrik's smoothed approximate functional equation, though with varying sophistication. Lavrik's method for controlling cancelation is often overlooked (method: use a parameter to control cancelation. Lagarias-Odlyzko mention it in their 1979 Artin L-function paper. I use it extensively in my computations).

1992 Berry-Keating gave a different smoothed approximate functional equation (for $\zeta(s)$).

Let

$$L(s) \sum_{n}^{\infty} \frac{b(n)}{s}$$

absolutely convergent n $\mathrm{Re}\, s > \sigma_1$

and et

$$\Lambda(s) \quad Q^s \prod_{j\ 1}^{a} \Gamma(\alpha_j s + \beta_j) L\ s)$$

$\alpha_j\ Q \in \mathbb{R}^+$

$\mathrm{Re}\, \beta_j \geq 0$

Assume that

1) $\Lambda(s)$ has meromorphic continuation to al of $\mathbb{C}$ w th simple poles at $s \quad s_\ell$ and corresponding residues $r \quad r_\ell$

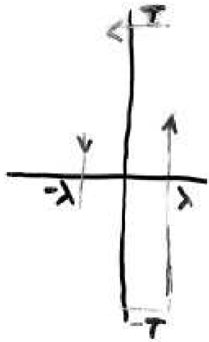
2) $\Lambda\ s) - w \overline{\Lambda(1-\bar{s})}$ for some $w \in \mathbb{C}$ $w \neq 0$

3) for any $c < d$ $\exists\, A\ c,d > 0$ such that

$$L(\sigma + it) \quad O(t^A) \qquad c \leq \sigma\ d$$

Consider

$$2\pi i \int_C \frac{\Lambda(z+s)\, g(z+s)}{z}\, dz$$



where

$g(z)$ s entire and satisfies
for s fixed and $\lambda < \mathrm{Re}\, z \le \lambda$,

$$\left| \frac{\Lambda(z+s)\, g(z+s)}{z} \right| \to 0 \quad \text{as} \quad |\mathrm{Im}\, z| \to \infty$$

λ and T are chosen big enough that all the poles $s_1 \quad s_\ell$ are contained in the rectangle, and also $\lambda > \sigma \quad \mathrm{Re}\, s$
and $\lambda > \sigma + \mathrm{Re}\, s - 1$

On the one hand we get

$$\Lambda(s) g(s) + \sum_{k=1}^{\ell} \frac{r_k\, g(s_k)}{s_k - s}$$

Now, as $T \to \infty$, $\int_{C_2}$ and $\int_{C_4} \to 0$

[redacted]

On C use the Dirichlet series for $\Lambda(z+s)$ and change order of summation and integration

On C_3 apply the functional equation and then the Dirichlet series

By Stirling's formula

$$|\Gamma(s)| \sim (2\pi)^{\frac{1}{2}} |s|^{\sigma - \frac{1}{2}} e^{-|t|\frac{\pi}{2}}, \quad \text{as } |t| \to \infty$$

So that $|\Lambda(s)|$ decreases exponentially fast as $|t| \to \infty$. On the other hand, with $g(z) = 1$, we can show that $f_1(s,n)$ and $f_2(1-s,n)$ start off comparatively large (even though they decrease exponentially fast in n).

So a <u>lot</u> of cancelation must occur!

The "standard" trick is to chose

$$g(z) = \delta^{-z}$$

with $|\delta| = 1$ and chosen to cancel out most of the exponentially small size of the Γ factors.

With $g(z) = \delta^{-z}$ we can write $f_1(s,n)$ and $f_2(1-s,n)$ explictly as incomplete integrals involving (unpleasant) special functions.

For example, when $a=1$ we have

$$\frac{\Gamma(v+u)}{u} = \int_0^\infty \Gamma(v,t)\, t^{u-1}\, dt\,, \quad \begin{array}{l}\mathrm{Re}\, u > 0 \\ \mathrm{Re}(v+u) > 0\end{array}$$

where

$$\Gamma(v,t) = \int_t^\infty e^{-x} x^{v-1}\, dx$$

is the incomplete gamma function. So, by Mellin inversion (with $g(z) = \delta^{-z}$)

$$f_1(s,n) = \delta^{-s}\, \Gamma\left(\alpha s + \beta, \left(n\delta/Q\right)^{\frac{1}{\alpha}}\right)$$

$$f_2(1-s,n) = \delta^{-s}\, \Gamma\left(\alpha(1-s) + \beta, \left(n/\delta Q\right)^{\frac{1}{\alpha}}\right)$$

ex 1

$$\pi^{\frac{s}{2}} \Gamma \ \frac{s}{2}) \xi(s \ \delta^{s}$$

$$\sum_{1}^{\infty} G(s/2 \ \ \pi n^2 \delta^2)$$

from pole at 0 from pole t_1

$$+\delta \sum_{n}^{\infty} G \ \ \frac{s}{2} \ \ \pi n^2 \delta^{-2}) \qquad s - \frac{\delta}{1-s}$$

$$G \ z\, w) \quad w^{z} \Gamma(z\, w) \ \int^{\infty} e^{wx} x^{z}\, d \quad \operatorname{Re} w > 0$$

ex 2 Dirichlet L f ctions

χ primitive conductor q

$\chi($

$$\left(\frac{q}{\pi}\right)^{s/} \Gamma(s/2) L \ s\ \chi) \delta$$

$$- \sum_{1}^{\infty} \chi(n \ G(s/2 \ \ \pi n^2 \delta^2 / q)$$

$$+ \frac{\tau(\chi)}{\delta q^{1/2}} \sum_{1}^{\infty} \chi \ n \ G\left(\frac{1-s}{2} \ \frac{\pi n^2}{\delta^2 q}\right)$$

$$\tau(\chi) - \sum^{q} \chi(m) e^{2\pi c\, m/q}$$

Notice n example if e take δ

$$^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s$$

$$s \qquad s + \pi^{\frac{s}{2}} \sum^{\infty} G\left(\frac{s}{2}\, \pi n^2\right) + \pi^{\frac{(1-s)}{2}} \sum^{\infty} G\left(\frac{1-s}{2}\, \pi n^2\right)$$

$$- - {}_{s} \qquad {}_{s} + \int_{1}^{\infty} \left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}}\right) \psi(x)\, dx$$

where

$$\psi(x) - \sum^{\infty} e^{\pi n^2 x}$$

Th s the expression Riemann discovered n his proof f the functional equation

To control decay of $\Gamma\left(\frac{s}{2}\right)$ we can take for s $\sigma + t$ with $t > 0$ $g(s)$ δ^{s}

$$\delta \begin{cases} & t < 2c/\pi \\ e^{c\left(\frac{\pi}{2} \; \frac{c}{t}\right)\frac{1}{2}} & t > 2c/\pi \end{cases}$$

t ke c 1 for example Larger c means faster convergence but als a proportional loss in accuracy)

$$G(z\,w) < \frac{e^{\mathrm{Re}\,w}}{\mathrm{Re}(w\ \ z+)} \qquad \text{if } \mathrm{Re}(w-z\ \ ) > 0$$

we get ex onential drop off roughly when

$$\mathrm{Re}\left(\pi n^2 \delta^2\right) >>$$

$$\text{But } \mathrm{Re}(\pi n^2 \delta^2) \quad \pi n^2\, \mathrm{Re}\,\delta^2 \quad \pi n^2 \frac{c}{t}$$

so terms starts to decay exponentially fast once

$$n >> \sqrt{\frac{t}{\pi c}}$$

constant ~ #digits log 0

sing properties of the incomplete gamma function one ca re te our

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s\ \ \delta^{s}$$

as a main sum $\sum^{N} \frac{1}{n^s} + \frac{\chi(s)}{n^{1-s}}$ + Remainder

$\chi(s)$

with $N \sim \sqrt{\frac{t}{2\pi}}$ and most f the computation spent th

Properties of the inco p te gamma function

$$\Gamma(z\ \omega)\ \int_{\omega}^{\infty} e^{t}\, t^{\ 1}\, dt \qquad \arg z < \pi$$

$$\gamma\ z\ \omega) = \int_{0}^{\omega} e^{t}\, t\ \ dt\ , \qquad \arg z < \pi$$
$$\mathrm{Re}\,\omega > 0$$

$$\Gamma(z\ \omega) + \gamma(z, \omega) - \Gamma(z)$$

$$\gamma^{*}(z\ \omega) = \frac{\omega^{\ z}\gamma(z\ \omega)}{\Gamma\ z)}$$ is entire in z and a

As before let

$$G(z\ \omega) = \omega^{\ z}\ \Gamma(z\ \omega)$$

$$g(z\ \omega)\ \ \omega^{\ z}\gamma(z\ \omega)$$

Integrate by parts

$$g(z\ w) \quad e^{w} \sum_{0}^{\infty} \frac{w^{j}}{(z)_{j+1}}$$

(z) $\quad f_j\ 0$

$z \quad z+j \quad f\ j>0$

Converges n cely say f $\mathrm{Re}\, z>0$
when $w \ll z$

Another series, useful for $w \ll$

$$g(z,w) \quad \sum_{0}^{\infty} \frac{(-)^n}{n} \frac{w^n}{z+n}$$

(not usef as w grows for the same reaso
that e^{x} $\sum \frac{(-x)^n}{n!}$ s not useful f x s big)

Continued fraction

$$g\ z\ w) \quad \cfrac{e^{w}}{z \quad \cfrac{z\ w}{z+ \ + \cfrac{w}{z+2 \quad \cfrac{(z+1)\cdot w}{z+3 + \cfrac{2\cdot w}{z+4 - (z+2)w}}}}}$$

works
well for $w \ll z$

Re a $\quad g(z\, w) = \underbrace{w^{z}\Gamma\, z}_{\text{eads to a main term}} - w \quad G(z\, w)$

For examp e for $\xi(s$,

$|w < |z$ is the reg on n the

fi st $G \; \frac{s}{2} \; \pi n^2 \delta^2$ s m

$$|\pi n^2 \delta \; < \left|\frac{s}{2}\right| \quad e \;\; n < \left.\frac{s}{2\pi}\right|^{1/2}$$

and n the se ond $G\left(\frac{-s}{2} \; \frac{\pi n^2}{\delta^2}\right)$ sum

$$\left|\frac{n\pi^2}{\delta^2}\right) < \left|\frac{1-s}{2}\right.$$

$$\text{e} \quad n < \left|\frac{1-s}{2\pi}\right|^{1/2}$$

$s \;\; 1/2 + it$ g ves a n n sum cons ting

of approx $t/_{2\pi}{}^{1/2}$ terms s me

as for R emann S ege for ζa

Integrate $G(z\,w$ by pa ts asymptotic

eries

$$G(z\,\omega)\quad \frac{e^{\omega}}{\omega}\sum_{0}^{M_{-}}\frac{(1-z)_{,}}{(\;\omega)^{,}} + \mathcal{E}_M(z\,\omega)$$

$$\mathcal{E}_M\; z\,\omega)_{-}\;\frac{(1-z)_M}{\;\;(\;\omega)}\;G(z\;M\quad)$$

works we $\quad f \quad \omega| \gg z$

In that region the following continued fraction works we

$$G(z\,\omega)\quad \cfrac{e^{\omega}}{\omega + \cfrac{z}{\;+\cfrac{}{\omega + \cfrac{2\;\;z}{1 + \cfrac{2}{\omega + 3\;z + \cfrac{3}{\omega +}}}}}}\qquad \text{arg} \;<\pi$$

Intermediate region $z \sim |w|$

s harder I that region $\Gamma(z\ w)$

undergoes a transition

One can use Nielsens expansion to

step through this region

$$\gamma(z\ w+d) \quad \gamma(z\ w) + w^{z}\ e \quad \sum_{j\ 0}^{\infty} \frac{(1-z)_j}{(w)^j}\left(-e^{-d}\ e_j(d)\right.$$

$$e_j\ d - \sum_{0}^{j} \frac{d^m}{m!}$$

$|d| < |w|$

This s very we suited for example

for L functions associated to modular form

since then we increment in eq step

Numerically unstable f d is big

Overcome this by taking many smaller teps

Not so well suited for degree L functions

One can also use Poisson summation method to handle $G(z,w)$ in the transition zone (soon)

or Temmes uniform asymptotics for $\Gamma(z\ w)$

$$Q(z\ w) \quad \frac{\Gamma(z\ w}{\Gamma\ z)}$$

$$\frac{1}{2}\mathrm{erfc}\left(\eta\sqrt{z/2}\right) + R_z(\eta)$$

$$\mathrm{erfc}(z) - \frac{2}{\sqrt{\pi}}\int_z^\infty e^{t^2}\,dt$$

$$\lambda = \frac{w}{z} \qquad \frac{1}{2}\eta^2 \quad \lambda \quad \log\lambda$$

sign η is chosen to be + for $\lambda >$

$$R_z(\eta) \quad \frac{e^{-\frac{1}{2}z\eta^2}}{\sqrt{2\pi z}} S_z(\eta)$$

$$S_z(\eta \sim \sum_{n\ 0}^{\infty} \frac{c_n(\eta}{z^n} \quad \text{as } z \to \infty$$

$$c_0(\eta) - \frac{1}{\lambda - 1} \quad \eta$$

(removable singularity at η 0 $(\lambda=1, z=a)$)

$$c_1\ \eta -$$

$$\eta^3 \quad \lambda\ 3^{-} \quad (\lambda\ 1)^2 \quad 2(\lambda\ )$$

$$\eta c_n\ \eta - \frac{d}{d\eta} c_n\ (\eta + \frac{\eta}{\lambda} \gamma_n \quad n >$$

where

$$\Gamma^*\ z) \sim \sum_{0}^{\infty} \frac{\ ^n\gamma_n}{z^n}$$

$$\Gamma^*(z) \quad \sqrt{\frac{z}{2\pi}} e^z z^z \Gamma(z)$$

$$\gamma \qquad \gamma \quad 12 \quad \gamma_2 \quad 288 \quad \gamma_3 \quad \frac{39}{51840}$$

When $a > 1$ we get much more painful incomplete integrals.

For example, if $\alpha_1 = \ldots = \alpha_a = 1/2$

$$f_1(s,n) = \delta^{-s} \int_{(n\delta/Q)^2}^{\infty} E_\alpha(t)\, t^{s/2+\mu-1}\, dt$$

$$f_2(1-s,n) = \delta^{-s} \int_{(n/\delta Q)^2}^{\infty} E_{\bar{\alpha}}(t)\, t^{\frac{1-s}{2}+\bar{\mu}-1}\, dt$$

where

$$\mu = \frac{1}{a} \sum_{1}^{a} \beta_j$$

$$E_\alpha(w) = \int_0^\infty \cdots \int_0^\infty \prod u_j^{\alpha_{j+1}-\alpha_j} e^{-w^{\frac{1}{a}}\left(\frac{1}{u_1} + \frac{u_1}{u_2} + \cdots \frac{u_{a-2}}{u_{a-1}} + u_{a-1}\right)} \frac{du_1}{u_1} \cdots \frac{du_{a-1}}{u_{a-1}}$$

(when $a=2$, $E_\alpha(w) = 2K_{\alpha_1-\alpha_2}(2w^{\frac{1}{2}})$, i.e. we get incomplete integrals of the K-Bessel function)

Now $\hat{h}$ is also rapidly decreasing.

$$\sum_{m \neq 0} \hat{h}\left(\frac{m}{\Delta}\right)$$

quickly becomes negligible as $\Delta \to 0$

So

$$\Delta \sum_{-\infty}^{\infty} h(\Delta m) \approx \hat{h}(0) \int_{-\infty}^{\infty} h(x)\,dx$$

i.e the Riemann sum makes for a great approx

This approach completely avoids the need for computing special functions. The most complicated thing we need to evaluate is the gamma function

We also have more freedom in choosing $g(t)$ since we don't need to know much (for ex series expansions) about $f(s,n)$.

Rather than work with the incomplete integrals we work directly with

$$f_1(s,n) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \prod \Gamma(\alpha_j(s+z)+\beta_j)\,\frac{g(s+z)}{z}\,(Q/n)^z\,dz$$

Ignoring the $g(s+z)$ for now, we note that the integrand is rapidly decreasing. It pays to be naive and compute the integral as a Riemann sum!

Rough idea Let $h: \mathbb{R}\to\mathbb{C}$ be smooth and rapidly decreasing. By Poisson summation

$$\Delta\sum_{-\infty}^{\infty} h(m\Delta) = \sum_{-\infty}^{\infty} \hat{h}\left(\frac{m}{\Delta}\right)$$

with

$$\hat{h}(u) = \int_{-\infty}^{\infty} h(x)\,e^{-2\pi i x u}\,dx$$

Substitute $z = v + iu$

$$f_1(s,n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod \Gamma(\alpha_j(s+v+iu)+\beta_j) \frac{g(s+v+iu)}{v+iu} (Q/n)^{v+iu}$$

We thus find (rearrange sums too)

$$\Lambda(s) g(s) \approx \sum_1^{l} \frac{r_k g(s_k)}{s - s_k} + Q^{s+v} \frac{\Delta}{2\pi} \sum_{m=-M}^{M} H_1(m\Delta) \sum_1^{N} \frac{b(n)}{n^{s+v+in}}$$
$$+ \omega \frac{Q^{1-s+v}}{2\pi} \Delta \sum_{m=-M}^{M} H_2(m\Delta) \sum_1^{N} \frac{\bar{b}(n)}{n^{1-s+v+im\Delta}}$$

where

$$H_1(u) = \prod \Gamma(\alpha_j(s+v+iu)+\beta_j) \frac{g(s+v+iu)}{v+iu} Q^{iu}$$

$$H_2(u) = \prod (\alpha_j(1-s+v+iu)+\bar{\beta}_j) \frac{g(s-v-iu)}{v+iu} Q^{iu}$$

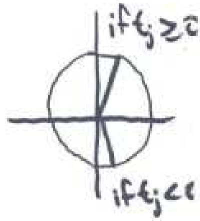
So, we compute $\Lambda(s)$ from the values of the Dirichlet series evaluated at equally spaced points to the right of s and $1-s$

Choice of g

Let

$$t_j = \mathrm{Im}(\alpha_j s + \beta_j)$$

$$\delta_j = \exp\left(i \,\mathrm{sgn}(t_j)\left(\frac{\pi}{2} - \varphi_j\right)\right)$$



$$\varphi_j = \begin{cases} \pi/2 & \text{if } |t_j| \le 2c/\pi \\ \dfrac{c}{|t_j|} & \text{if } |t_j| > 2c/\pi \end{cases}$$

$c > 0$ is a free parameter. Larger c means faster conv. but more loss of precision.

Set

$$g(w) = \underbrace{\exp\left(A(w-s)^2\right)}_{\text{allows us to cut down on the domain of integration.}} \underbrace{\prod_{j=1}^{a} \delta_j^{-(\alpha_j w + \beta_j)}}_{\text{controls the } \prod \Gamma(\alpha_j s + \beta_j)}$$

A can't be large, though, since we need the fourier transform of this factor to have small "support".

$$A = \frac{1}{16\pi^2 \log 10} \frac{1}{\text{Digits}}$$

$$\Delta = \frac{2\pi}{\log 10} \frac{v}{\text{Digits}}$$, v is a free parameter If it is too large or too small we lose precision (with v=.6, Digits=12, we have $\Delta = .136$)

$$M = \frac{2 \log^2(10)\, \text{Digits}^2}{v}$$

$$N = O\left(Q(|t|+1)^{\sum \alpha_j} \right)$$

the Q that appears in the functional equation: For $L(s,\chi)$, $Q = \left(\frac{q}{\pi}\right)^{1/2}$, q conductor of χ