Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Statistics of low-lying zeros of L-function and random matrix theory III

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PROOF OF GAUDIN'S LEMMA

Let π_M be the symmetric group on $\{1, \ldots, M\}$. Then,

$$\det_{M \times M} \big(f(\theta_j, \theta_k) \big)$$

$$\sum_{\sigma \in \pi_M} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}).$$

If $\sigma M \neq M$, then

$$\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M$$

$$= \prod_{\substack{j=1 \\ \sigma j \neq M}}^{M-1} f(\theta_j, \theta_{\sigma j}) \int_J f(\theta_{\sigma^{-1} M}, \theta_M) f(\theta_M, \theta_{\sigma M})\, d\theta_M$$

$$= f(\theta_{\sigma^{-1} M}, \theta_{\sigma M}) \prod_{\substack{j=1 \\ \sigma j \neq M}}^{M-1} f(\theta_j, \theta_{\sigma j})$$

For a permutation $\sigma \in \pi_M$ with $\sigma M \neq M$ define a permutation $\sigma' \in \pi_{M-1}$ by

$$\sigma' j = \begin{cases} \sigma j & \text{if } \sigma j \neq M \\ \sigma M & \text{if } \sigma j = M \end{cases}$$

Then the above may be reexpressed as

$$\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M = C \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}).$$

Clearly, each permutation σ' arises from $(M-1)$ different σ. Note also that $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$. Thus, we have

$$\begin{aligned}
&\int_J \sum_{\substack{\sigma \in \pi_M \\ \sigma M \neq M}} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M \\
&\qquad = -(M-1)C \sum_{\sigma' \in \pi_{M-1}} \operatorname{sgn}(\sigma') \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}) \\
&\qquad = -(M-1)C \det_{M-1} \big(f(\theta_k, \theta_j) \big).
\end{aligned}$$

Now consider the σ for which $\sigma M = M$; now let σ' be defined by

$\sigma' j = \sigma j$ for $j \le M-1$. Then, for these σ, we have

$$\begin{aligned}
&\int_J \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M \\
&\qquad = \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma j}) \int_J f(\theta_m, \theta_M)\ d\theta_M \\
&\qquad = D \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j}).
\end{aligned}$$

These σ' have the same sign as the σ they came from. Therefore,

$$\int_J \sum_{\substack{\sigma \in \pi_M \\ \sigma M = M}} \operatorname{sgn}(\sigma) \prod_{j=1}^{M} f(\theta_j, \theta_{\sigma j}) d\theta_M$$

$$= D \sum_{\sigma' \in \pi_{M-1}} \operatorname{sgn}(\sigma') \prod_{j=1}^{M-1} f(\theta_j, \theta_{\sigma' j})$$

$$= D \det_{M-1} \big(f(\theta_k, \theta_j) \big).$$

Combining the two cases we obtain the Lemma.