Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Toeplitz determinants & connections to random matrices III

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TWO LIMIT THS

φ smooth

$$D_n(\varphi) \sim G(\varphi)^n E(\varphi)$$

$$E(\varphi) \quad \det\left(T(\varphi)T(\varphi^{-1})\right)$$

$$\exp\left(\sum_1^\infty k s_k s_{-k}\right)$$

Fisher-Hartwig

$$D_n(\varphi) \sim G(\varphi)\, n^{\Omega} E^*(\varphi)$$

$$\Omega \quad \sum^{R} \alpha_j^2 \beta_j^2$$

In order to understand the connections of Szego Limit Ths and Fisher-Hartwig symbols we need two formulas.

Formula 1.

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(t-\mu)^2}{2\sigma^2}} e^{ixt}\, dt$$

$$= e^{ix\mu} \; e^{-x^2\sigma^2}$$

We can read off μ and σ^2 from the transform of a Gaussian distribution

Formula 2

$$\frac{1}{N}\int \int \det\left(f_j(x_k)\right)\det\left(g_j(x_k)\right)dx_1\, dx_N$$

$$= \det\left(\int f_j(x)\, g_j(x)\, dx\right)_{j,k\ 1,\ N}$$

This says that anything of the form

$$\frac{1}{N!}\int_{\pi}^{\pi}\int_{-\pi}^{\pi}\prod_{i<j}\left|e^{i\theta_i}\quad e^{i\theta_j}\right|^2\prod_{i=1}^{N}\varphi(e^{i\theta_i})$$

$$\times\, d\theta_1 \quad d\theta_N$$

$$= \det\left(\int_{-\pi}^{\pi}\varphi(\theta)\, e^{ik\theta}\, e^{-ij\theta}\, d\theta\right)_{kj=0}^{N-1}$$

$f_j(x)\ \ \varphi(x)x^{j-1},\ g_j(x)\ \ \ x^{-j+1}$

So now let us consider a random variable (linear statistic) of the form $\sum_{j=1}^{N} f(e^{i\theta_j})$

From Probability Theory we know the Fourier transform of the distribution should be

$$g(\lambda) = \frac{1}{N}\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i\lambda \sum_{j=1}^{N} f(e^{i\theta_j})}$$

$$\times \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_N$$

$$= D_N(\varphi) \qquad \varphi \qquad e^{i\lambda f}$$

By the SSLT,

$$D_N(\varphi) \sim G(\varphi)^N \exp\left(\sum_1^\infty k s_k s_{-k}\right).$$

$$\varphi(e^{i\theta}) = e^{i\lambda f(e^{i\theta})}$$

$$G(\varphi)^N = e^{i\lambda \frac{N}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,d\theta}$$

$$\exp\left(\sum_1^\infty k s_k s_{-k}\right)$$

$$= \exp\left(-\lambda^2 \sum_1^\infty k f_k f_{-k}\right)$$

which says that in the limit we have a gaussian with $\mu = \frac{N}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,d\theta$

and $\sigma^2 = \sum_1^\infty k f_k f_{-k}$

$$= \sum_1^\infty k |f_k|^2 .$$

But what happens for a discontinuous f, say $f(e^{i\theta})$ $\chi_I(e^{i\theta})$

$$= \begin{cases} 1 & \text{if } e^{i\theta} \in I \\ 0 & \text{otherwise} \end{cases}$$

So we are led to consider

$$\varphi(e^{i\theta}) = e^{i\lambda 2n \chi_I}$$

This is a Fisher Hartwig symbol.

For $\varphi_{0,\beta} = e^{i\beta(\theta - \pi)}$

$$0 < \theta < 2\pi$$

$$\beta \qquad \frac{1}{2\pi i} \log\left(\frac{\varphi(1-)}{\varphi(1+)}\right)$$

For $e^{i\lambda 2\pi \chi_I}$

with I arc from

$e^{i\alpha}$ to $e^{i\alpha}$

we have two jumps

$\lambda \qquad \frac{1}{2\pi i} \log\left(\frac{1}{e^{i\lambda 2\pi}}\right)$

$- \quad \lambda \qquad (\text{for } e^{i\alpha})$

$\lambda_2 = \frac{1}{2\pi i} \log\left(e^{i\lambda 2\pi}\right)$

$= \lambda$

So we check

$\varphi_{0,+\lambda,\alpha} \qquad \varphi_{0,-\lambda,-\alpha}$

$e^{+i\lambda(\theta-\pi-\alpha)} \qquad e^{i\lambda(\theta-\pi+\alpha)}$

$\alpha < \theta < 2\pi + \alpha \qquad \alpha < \theta < 2\pi \quad \alpha$

$$e^{i\lambda 2\pi x_I}$$

$$= e^{2i\lambda\alpha} \quad \varphi_{0\ \lambda,\alpha} \quad \varphi_{0,\ \lambda,\ \alpha}$$

If we apply F H

we have an answer

of the form for

the distribution transform

$$e^{2i\lambda N\alpha} \quad N^{\ 2\lambda^2} \ (2 \quad 2\cos 2\alpha)^{\lambda^2}$$

$$\times \quad G(1 \quad \lambda)^2 G(1+\lambda)^2$$

Write $N^{\ 2\lambda^2}$

$$e^{-2\lambda^2 \log N},$$

so $\sigma^2 \qquad +2\log N$

Hence to get a finite limit we should begin with the variable

$$\frac{1}{\sqrt{2\log N}} \sum_{j=1}^{N} \left(\chi_I(e^{i\theta_j}) - 2\alpha\right)$$

then Fourier transform tends to (from F H)

to e^{λ^2},

as long as $\left|\frac{\lambda}{\sqrt{2\log N}}\right| \le c < \frac{1}{2}$

So for CUE

we need $\det T_n(e^{i\lambda f})$

Think of $T_n(\varphi)$

as $P_n(\varphi(P_n f)^\wedge)^\vee$

For GUE

analogue is $P_\alpha(\varphi(P_\alpha f)^\vee)^\vee$

$g^\vee$ is the Fourier

transform of g

This is Wiener-Hopf

operator $W_\alpha(\varphi)$

and we need for linear statistics

$\det(I + W_\alpha(\varphi))$

$\varphi \quad e^{i\lambda f}$,

If $\varphi = e^{b} - 1$, then

$$\det(I + W_\alpha(b))$$

$$\sim \exp\left(\frac{\alpha}{2\pi} \int_{-\infty}^{\infty} b(x)\,dx + \int_0^{\infty} x\, \hat{b}(x)\hat{b}(-x)\,dx \right)$$

Note: Asymptotics for the analogues of Fisher-Hartwig have finally been done.

For Laguerre, the operator is

$$P \; H_\nu M_\varphi H_\nu P_\alpha$$

H Hankel transform

$$H_\nu(f)(x) \quad \int_0^\infty \sqrt{tx}\; J_\nu(tx) f(t)\,dt$$

If $\varphi = e^b \quad 1$, then

$$\det(I + B_\alpha(\varphi))$$

$$\sim \exp\left(\alpha \hat{b}(0) \quad \frac{\nu}{2} b(0) + \frac{1}{2}\int_0^\infty x \hat{b}(x)^2 dx\right)$$

as $\alpha \to \infty$

Finally for the edge we are led to

$$f(x/\alpha)\int_0^\infty A(x+z)A(z+y)\,dz$$

the Airy Operator

$$\det(I + A_\alpha(f))$$

$$\sim \exp\{c\,\alpha^{3/2} + c_2\}$$

$$c_1 \quad \frac{1}{\pi}\int_0^\infty \sqrt{x}\,\log(1+f(x))\,dx$$

$$c_2 \quad \frac{1}{2}\int_0^\infty x\,G(x)^2\,dx$$

$$G(x) \quad \frac{1}{2\pi}\int_{-\infty}^\infty e^{xy}\log(1+f(-y^2))\,dy$$

There are also Fisher-Hartwig analogues for

$$T_n(\varphi) + H_n(\varphi) \qquad M_n(\varphi)$$

$$= (a_{j} + a_{+j+})_{j\,0}^{N\;1}$$

$$\det M_n(\varphi) \sim G(\varphi)^n n^{\Omega} E^{*}(\varphi)$$

Ω depends on the location of the singularity

① for $\varphi = \varphi_{0\,\beta\,0}$ $\quad \Omega \quad \frac{3\beta}{2} \quad \frac{\beta}{2}$

$\varphi = \varphi_{0,\beta\,\pi}$ $\quad \Omega = -\frac{3\beta^2}{2} + \frac{\beta}{2}$

$0 < \theta_0 < \pi$ $\quad \varphi = \varphi_{0,\beta,\theta_0}$ $\quad \Omega = \beta^2$

$\varphi \quad \varphi_{0\,\beta\,\theta_0}\, \varphi_{\,\beta\,\theta} \qquad \Omega \quad \beta$