Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Moments of L-functions III

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MEAN VALUES OF DIRICHLET POLYNOMIALS AND APPLICATIONS

I. Mean Values of Dirichlet Polynomials

Let

$$A(s) = A_N(s) = \sum_{n=1}^{N} a_n n^{-s}$$

be a Dirichlet polynomial of length N. Let $s = \sigma + it$

Theorem. (Classical Mean Value Theorem)

$$\int_0^T \Big|\sum_{n=1}^{N} a_n n^{-s}\Big|^2 dt = \Big(T + O(N \log N)\Big) \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}$$

The more precise version of H. L. Montgomery and R. C. Vaughan is

Theorem. (Montgomery–Vaughan)

$$\int_0^T \Big|\sum_{n=1}^{N} a_n n^{-s}\Big|^2 dt = \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma} \Big(T + O(n)\Big)$$

From this we see that if $N = o(T)$, then

$$\int_0^T \sum_{n=1}^{N} a_n n^{-\sigma-it}|^2 dt \sim T \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}$$

On the other hand, if $N \gg T$ the O–term dominates and we have

only

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-\sigma-it}|^2 dt \ll N \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma}$$

Is this really the size of the mean when N is larger than T?

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V gh mea form: ive

$$\int^{T} \sum^{?} n \quad dt \quad \sum^{N} \frac{1}{n}\left(T + O\, n\right)$$

$$T \quad o \quad ? \; + O\,1 \quad + O\, N$$

$$\begin{cases} 1 + o\,1 \quad T \;\text{og}\, N & N \quad O\,T \\ O\, N & ? \; > T' \quad \alpha > 1 \end{cases}$$

W can ev :te the mea lternati wa by th form

$$\zeta \qquad \sum_{?} n \quad + \;^{N} \; 1 + O\, ?$$

V :t! σ tl

$$\int^{T} \sum^{?} n \qquad dt$$

$$\int^{T} \quad \frac{1}{2} + \qquad + \frac{N^{\frac{1}{2} \; it}}{\frac{1}{2} - it} + O\, N \qquad dt$$

We have

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt \quad \boldsymbol{T}\log \boldsymbol{T}$$

$$\int_0^T \left|\frac{N^{\frac{1}{2}-it}}{1/2-it}\right|^2 dt - N\int_0^T \frac{1}{1/4+t^2}dt \qquad N$$

and

$$\int_0 N \quad dt - \boldsymbol{T}\ N$$

Therefore

$$\int_0^T \sum \qquad {}^{\boldsymbol{it}}\Big|^2 dt \quad \pi N$$

for $N \gg T^\alpha$ with $>$

So the $O-$ term can not be reduced this case

It still makes sense to ask

Question. Is there a useful precise formula for

$$\int_0^T \sum_{n=1}^{N} a_n n^{-s}|^2 dt$$

when $N = T^\alpha$, $\alpha > 1$?

To answer this recall the proof of the Classical Mean Value Theorem.

Squaring out and integrating, we have

$$\int_0^T |\sum_{n=1}^{N} a_n n^{-\sigma-it}|^2 dt$$

$$\int_0^T \sum_{n=1}^{N} \quad a_n \overline{a}_m n^{-\sigma-it} m^{-\sigma+it} dt$$

$$\sum_{n=1}^{N} \sum_{m=1}^{N} \frac{a_n \overline{a}_m}{(nm)^\sigma} \int_0^T (m/n)^{it} \, dt$$

$$T \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\sigma}} + \sum_{\substack{1 \le m,n \le N \\ m \ne n}} \frac{a_n \overline{a}_m}{(nm)^\sigma} \left(\frac{e^{iT \log(m/n)} - 1}{i \log(m/n)} \right)$$

The second sum consists of "off–diagonal" terms and is

$$\ll \sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{|a_n a_m|}{(nm)^{\sigma}|\log(m/n)|}$$

$$\le \frac{1}{2} \sum_{\substack{1\le m,n\le N\\ m\ne n}} \left(\frac{|a_n|^2}{n^{2\sigma}} + \frac{|a_m|^2}{m^{2\sigma}}\right)\frac{1}{|\log(m/n)|}$$

$$\sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{|a_n|^2}{n^{2\sigma}|\log(m/n)|}$$

$$\sum_{1\le n\le N} \frac{|a_n|^2}{n^{2\sigma}}\left(\sum_{\substack{1\le m\le N\\ m\ne n}} \frac{1}{|\log(m/n)|}\right)$$

The inner sum is

$$\ll \left(\sum_{\substack{m\le N\\ |m-n|\le n/2}} + \sum_{\substack{m\le N\\ n/2<|m-n|}} \right)\frac{1}{|\log(m/n)|}$$

$$\ll \sum_{1\le h\le n/2} \frac{1}{|\log((n\pm h)/n)|} + \sum_{\substack{m<n/2 \text{ or}\\ 3n/2<m\le N}} \frac{1}{|\log(m/n)|}$$

$$\ll \sum_{1\le h\le N} \frac{n}{h} + \sum_{1\le m\le N} 1$$

$$\ll N\log N + N \ll N\log N\,.$$

Hence, the off–diagonal terms are

$$\ll N \log N \sum_{1\le n\le N} \frac{|a_n|^2}{n^{2\sigma}}$$

We therefore obtain

$$\int_0^T \Big|\sum_{n=1}^N a_n n^{-\sigma-it}\Big|^2 dt = \big(T + O(N\log N)\big) \sum_{n=1}^N \frac{|a_n|^2}{n^{2\sigma}}$$

Clearly, if we want a better formula, we need to estimate the off–diagonal terms more carefully. Returning to our initial expression for these terms, we see that

$$\sum_{\substack{1\le m,n\le N\\ m\ne n}} \frac{a_n\overline{a}_m}{(nm)^\sigma}\left(\frac{e^{iT\log(m/n)}-1}{i\log(m/n)}\right)$$

$$2Re \sum_{1\le n<m\le N} \frac{a_n\overline{a}_m}{(nm)^\sigma}\left(\frac{e^{iT\log(m/n)}-1}{i\log(m/n)}\right)$$

$$= 2Re \sum_{1\le n<N}\ \sum_{1\le h\le N-n} \frac{a_n\overline{a}_{n+h}}{(n(n+h))^\sigma}\left(\frac{e^{iT\log((n+h)/n)}-1}{i\log((n+h)/n)}\right)$$

$$2Re \sum_{1\le h<N}\ \sum_{1\le n\le N-h} \frac{a_n\overline{a}_{n+h}}{n^{2\sigma}}(1+h/n)^\sigma\left(\frac{e^{iT\log(1+h/n)}-1}{i\log(1+h/n)}\right)$$

To get a feel for this, consider only the terms for which $h/n < 1/2$.

In these we can approximate

$$\log(1 + h/n) \quad \text{by} \quad h/n$$

and

$$(1 + h/n)^{\sigma} \quad \text{by} \quad 1.$$

This part of the sum therefore contributes approximately

$$2Re \sum_{1 \le h < N} \sum_{2h < n \le N-h} \frac{a_n \overline{a}_{n+h}}{n^{2\sigma-1}} \left(\frac{e^{iTh/n} - 1}{ih} \right)$$

To simplify further, consider only the terms with $Th < n/2$, say. Then

$(e^{iTh/n} - 1)/ih \approx T/n$, so these terms are approximately

$$2T\, Re \sum_{h \neq 0} \sum_{n} a_n \overline{a}_{n+h} n^{-2\sigma}$$

We can treat this if we have good estimates for the sums

$$\sum_{n=1}^{N} a_n \, \overline{a}_{n+h} \, .$$

II. The Mean Value of Long Dirichlet Polynomials.

Assumptions.

1. (Normalization)

$$a_n \ll n^\epsilon$$

2. There is a function $M(x)$ and a $0 < \theta < 1$ such that

$$\sum_{n \le x} a_n = M(x) + O(x^\theta),$$

and $M'(x) \ll x^\epsilon, M''(x) \ll x^{-1+\epsilon}$

3. There is a function $M(x,h)$, a $0 < \phi < 1$, and an $0 < \eta < 1$

such that

$$\sum_{n \le x} a_n \overline{a}_{n+h} = M(x,h) + O(x^\phi)$$

uniformly for $h \le x^\eta$, and

$$M'(x,h) \ll (hx)^\epsilon$$

Instead of

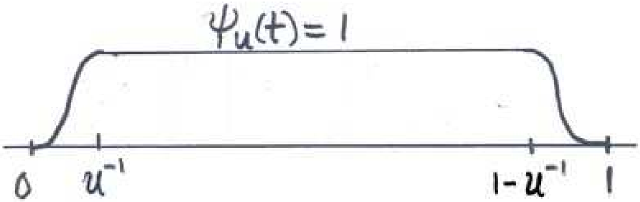
$$\int_0^T \sum a_n n \ {}^{s}|^2 dt$$

$$\int_0^T \sum a_n n \qquad \int_1^N M'(x)x \quad dx|^2 dt$$

is more natural for applications. Also,

$$\int^\infty \Psi_U(\frac{t}{T})|\sum_{n=1} a_n n \qquad \int M'(x)x \quad dx|^2 dt$$

is much simpler technically, where $\Psi_U(t/T)$ is the weight function



$$\Psi_U(v) \qquad\qquad \text{for}|v| <$$

$$\Psi_U^{(j)}(t) \ll \log^j T$$

$$U - \log^A T$$

Theorem. (Goldston–Gonek) *Let $\epsilon > 0$ be arbitraily small, $\sigma < 1$, and θ, ϕ, η as before. Then for $T \ll N \ll T^{(1-\epsilon)/(1-\eta)}$ we have*

$$\int_{-\infty}^{\infty} \Psi(\frac{t}{T}) \Big| \sum_{n=1}^{N} a_n n^{-s} \quad \int_1^N M'(x) x^{-s} dx \Big|^2 dt$$

$$= \hat{\Psi}(0) T \sum_{n \le N} |a_n|^2 n^{-2\sigma}$$

$$+ 4\pi (\frac{T}{2\pi})^{2-2\sigma} Re \int_{T/2\pi N}^{\infty} \Big(\sum_{1 \le h \le 2\pi N v/T} M'(\frac{hT}{2\pi v}, h) h^{1-2\sigma} \Big) \frac{\hat{\Psi}(v)}{v^{2-2\sigma}} dv$$

$$- 4\pi (\frac{T}{2\pi})^{2-2\sigma} Re \int_{T^{\epsilon}/2\pi N}^{\infty} \Big(\int_0^{2\pi N v/T} M'(\frac{uT}{2\pi v}) \Big|^2 u^{1-2\sigma} du \Big) \frac{\hat{\Psi}(v)}{v^{2-2\sigma}} dv$$

$$+ O(N^{1-2\sigma+max(\theta,\phi)+5\epsilon}) + O(N^{2-2\sigma+5\epsilon} T^{-1}) + O(N^{2\epsilon}).$$

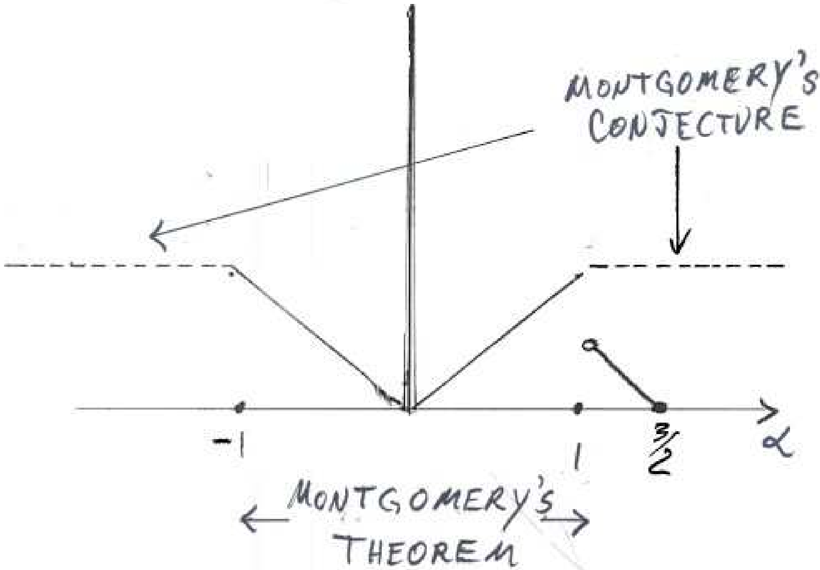
III. Application: A Lower Bound for $F(\alpha)$

Recall that

$$F(\alpha) = (\frac{T}{2\pi}\log T)^{-1} \sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')} \frac{4}{4+(\gamma-\gamma')^2}$$

$F(\alpha)$ is even and $F(\alpha) \geq 0$. We also write

$$F(x,T) = \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} \frac{4}{4+(\gamma-\gamma')^2}$$



Theorem. (Goldston–Gonek–Ozluk–Snyder) *Assume the Generalized Riemann Hypothesis. Then*

$$F(\alpha) \geq \frac{3}{2} \quad |\alpha| \quad \epsilon$$

uniformly for $1 \leq |\alpha| \leq 3/2 \quad 2\epsilon$ *and* $T \geq T_0(\epsilon)$.

Sketch

We use the following explicit formula.

$$-2 \sum_{0<\gamma\leq T} x^{i(\gamma-t)} \frac{1}{1+(t-\gamma)^2}$$

$$= x^{-1}\left(\sum_{n\leq x} \Lambda(n) n^{\frac{1}{2}-it} \quad \int_1^x u^{\frac{1}{2}-it} du \right)$$

$$+ x\left(\sum_{n>x} \Lambda(n) n^{-\frac{3}{2}-it} \quad \int_x^\infty u^{-\frac{3}{2}-it} du \right) + E(x,t)$$

Integrate the modulus squared of both sides. The left–hand side is

$$\int_0^T \Big|2 \sum_{0<\gamma\le T} x^{i(\gamma-t)} \frac{1}{1+(t-\gamma)^2}\Big|^2 dt$$

$$= 2\pi \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} \frac{4}{4+(\gamma-\gamma')^2} + E(x,T)$$

$$= 2\pi\, F(x,T) + E(x,T)$$

We equate this with the mean modulus squared of the right–hand side

$2\pi\, F(x,T)$

$$\int_0^T \Big|x^{-1}\Big(\sum_{n\le x}\Lambda(n)n^{\frac{1}{2}-it} \quad \int_1^x u^{\frac{1}{2}-it}du\Big)$$

$$+x\Big(\sum_{n>x}\Lambda(n)n^{-\frac{3}{2}-it} \quad \int_x^\infty u^{-\frac{3}{2}-it}du\Big)\Big|^2 dt$$

$$+E(x,T)\,.$$

Case 1. $x = T^\alpha\,,\ \alpha < 1$

Classical MVT $\Longrightarrow$ Montgomery's Theorem

Case 2. $x = T^{\alpha}, \ \alpha > 1$

Long MVT and the Twin Prime Conjecture

(i.e.,good estimates for $\sum_{n \leq y} \Lambda(n)\Lambda(n+h)$)

$\implies$ Montgomery's Conjecture

We have no proof of the twin prime conjecture, but we can approximate

the $\Lambda(n)$'s by the coefficients

$$\lambda_Q(n) = \sum_{q \leq Q} \frac{\mu^2(q)}{\phi(q)} \sum_{\substack{d|q \\ d|n}} d\mu(d) \ ,$$

and we do have an analogue of twin primes for this.

We saw that

$$2\pi\, F(x\ T)$$

$$-\int_0^T \Big| x\ \Big(\sum_{n<x} \Lambda(n) n^{2-it}\ \int^x u^2\ du\Big)$$

$$+\, x\Big(\sum \Lambda(n) n\ \int_x^\infty\ {}^{it} du\Big)\Big|^2 dt$$

$$+\, E(x\ T)$$

We write this

$$2\pi\, F(x\ T) - \int_0^T \left|\mathbb{A}(x\ t) + \mathbb{A}\ (x\ t)\right|^2 dt + E(x\ \boldsymbol{T})$$

Let $\mathbb{A}_Q(x\ t)$ and $\mathbb{A}^*_Q(x\ t)$ be the same as $\mathbb{A}(x\ t)$ and $\mathbb{A}^*(x\ t)$ respectively, but with the $\lambda_Q(n)$' replacing the $\Lambda(n)$'

We have

$$0 \leq \int_{-\infty}^{\infty} \Psi_U(\frac{t}{T}) \Big| \big(\mathbb{A}(x,t) + \mathbb{A}^*(x,t)\big) \quad \big(\mathbb{A}_Q(x,t) + \mathbb{A}_Q^*(x,t)\big) \Big|^2 dt\,.$$

This implies

$$2Re \int_{-\infty}^{\infty} \Psi_U(\frac{t}{T}) \Big(\mathbb{A}\overline{\mathbb{A}_Q} + \mathbb{A}^*\overline{\mathbb{A}_Q^*} + \mathbb{A}\overline{\mathbb{A}_Q^*} + \mathbb{A}^*\overline{\mathbb{A}_Q} - \mathbb{A}_Q\overline{\mathbb{A}_Q^*} \Big) dt$$

$$\int_{-\infty}^{\infty} \Psi_U(\frac{t}{T}) \big(\mathbb{A}_Q\overline{\mathbb{A}_Q} + \mathbb{A}_Q^*\overline{\mathbb{A}_Q^*}\big) dt$$

$$\leq \int_{-\infty}^{\infty} \Psi_U(\frac{t}{T}) \big| \big(\mathbb{A}(x,t) + \mathbb{A}^*(x,t)\big) \big|^2 dt$$

$$= 2\pi\, F(x,T)\,.$$

The necessary coefficient correlation sums needed to estimate long mean–values are:

$$\mathbb{A}\mathbb{A}_Q, \qquad \mathbb{A}^*\overline{\mathbb{A}_Q^*} \longleftrightarrow \sum_{n \leq y} \Lambda(n)\lambda_Q(n+h)$$

and

$$\mathbb{A}_Q\overline{\mathbb{A}_Q}, \qquad \mathbb{A}_Q^*\overline{\mathbb{A}_Q^*} \longleftrightarrow \sum_{n \leq y} \lambda_Q(n)\lambda_Q(n+h)$$

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rali *ed* *R* *eman* *n* *Hypothesis* *Let* Q $^{\delta}$ *with* $/4 < \delta \le 1/$

Then

$$\sum_{n\le y} \Lambda\ n\ \lambda_Q \quad + h \qquad nd \qquad \sum \lambda_Q\ n\ \lambda_Q\ n + h$$

re both $h\ y + O\ y^{\ +\delta+}$ *uniformly fo* $< h \le y$ *wh re*

$$c(h \quad \begin{cases} \prod_{p>} \quad - \frac{1}{(p-1)^2} \prod_{p} \quad \frac{p-1}{p-2} & f\ h \quad even \\ 0 & f\ h\ is\ odd \end{cases}$$

Applyi tl to te ms t. X T^{α} o δ optima

ctio α ea to

$$F\ \alpha\ T \ge \frac{3}{2} \quad \alpha$$

for $1 < \alpha \le$

Application: The 6th and 8th Moments of the Zeta Function

Hardy and Littlewood (1918):

$$\int_0^T |\zeta(1/2+it)|^2\, dt \sim T\log T \qquad (\text{as} T \to \infty)$$

Ingham (1926):

$$\int_0^T |\zeta(1/2+it)|^4\, dt \sim \frac{1}{2\pi^2} T\log^4 T$$

No analogous formula has been proved for any other moment

$$I_k(T) = \int_0^T |\zeta(1/2+it)|^{2k}\, dt$$

J B. Conrey and A. Ghosh conjectured

Conjecture. (J. B. Conrey–A. Ghosh) *As* $T \to \infty$,

$$\int_0^T |\zeta(1/2+it)|^6\, dt \sim \frac{42}{9!}\prod_p \left(\left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\right) T\log^9 T$$

Later, J. B. Conrey and I made the

Conjecture. **(J. B. Conrey–S.Gonek)** *As* $T \to \infty$,

$$\int_0^T |\zeta(1/2+it)|^8 \, dt$$

$$\frac{24024}{16!} \prod_p \left(\left(1-\frac{1}{p}\right)^9 \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)\right) T \log^{16} T.$$

At the same time, J. Keating and N. Snaith conjectured the size of all the moments $I_k(T)$ based on the moments of characteristic polynomials of random matrices.

We sketch our argument for the 8th moment conjecture. It gives the 2nd, 4th, and 6th moment asymptotics as well.

We begin with a discussion of the approximate functional equation.

For $s = \sigma + it$ and $\sigma > 1$, $\zeta^k(s)$ has the Dirichlet series expansion

$$\begin{aligned}\zeta^k(s) &= \prod_p \left(1 - p^{-s}\right)^{-k} \\ &= \prod_p \left(1 + \frac{d_k(p)}{p^s} + \frac{d_k(p^2)}{p^{2s}} + \cdots\right) \\ &= \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}\end{aligned}$$

where $d_k(p^j) = (-1)^j \binom{-k}{j}$ is the kth divisor function.

This series does not converge if $\sigma \le 1$, but we can approximate $\zeta^k(s)$ in this region by an expression of the form

$$\zeta(s)^k = \sum_{n=1}^{N} \frac{d_k(n)}{n^s} + \chi(s)^k \sum_{n=1}^{M} \frac{d_k(n)}{n^{1-s}} + \mathbb{E}_k(s)$$

This is an approximate functional equation.

We write this as

$$\zeta(s)^k = \mathbb{D}_{k,N}(s) + \chi(s)^k \mathbb{D}_{k,M}(1-s) + \mathbb{E}_k(s)\,,$$

where

$$\mathbb{D}_{k,N}(s) = \sum_{n=1}^{N} \frac{d_k(n)}{n^s}\,,$$

$$\mathbb{E}_k(s),$$

is an error term,

$$MN = \left(\frac{t}{2\pi}\right)^k$$

and

$$\chi(s) = \pi^{s-1/2}\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$$

is the factor from the functional equation for the zeta function,

$$\zeta(s) = \chi(s)\zeta(1-s)\,.$$

Taking $= 1/2 +$ we find that

$$\zeta(\tfrac{1}{2} + it)^k = \mathbb{D}_k \; (\tfrac{1}{2} + it) + \chi(\tfrac{1}{2} + it)^k \mathbb{D}_{k,M}(\tfrac{1}{2} \quad it) + \mathbb{E}_k(\tfrac{}{2} + it)$$

Assuming that $\mathbb{E}_k(1/2 + it)$ may be ignored we obtain

$$\int_T^{2T} |\zeta(1/2 + it)|^{2k} \, dt$$

$$\int_T^{2T} |\mathbb{D}_{k,N}(1/2 + it)|^2 \, dt + \int_T^{2T} |\mathbb{D}_{k,M}(1/2 + it)|^2 \, dt$$

$$+ 2Re \int_T^{2T} \chi(1/2 \quad it)^k \mathbb{D}_{k,N}(1/2 + it) \mathbb{D}_{k,M}(1/2 + it) \, dt$$

There is reason to believe that the cross term smaller than the main

term and that $MN = (t/2\pi)^k$ may be replaced by $MN = (T/2\pi$

Thus we expect that

$$\int_T^{2T} |\zeta(1/2 + it)|^{2k} \, dt$$

$$\int_T^{2T} |\mathbb{D}_{k,N}(1/2 + it)|^2 \, dt + \int_T^{2T} |\mathbb{D}_{k,M}(1/2 + it)|^2 \, dt$$

where $MN = (T/2\pi)$ and $M \; N > 1$

We can prove this when $k = 1$ or $k = 2$, provided that M and N are both $\ll T$.

When $k \geq 3$, the known bounds for $\mathbb{E}_k(s)$ are too large and it is difficult to show that the cross term really is small. (However, it might be possible to overcome these problems by appealing to a more complicated form of the approximate functional equation first developed by A. Good.)

Our problem now is to determine an asymptotic estimate for the mean square of the Dirichlet polynomials $\mathbb{D}_{k,N}(1/2+it)$ and $\mathbb{D}_{k,M}(1/2+it)$.

Montgomery and Vaughan mean value theorem

$$\int_T^{2T} \sum^{N} a(n)n^{it}|^2 dt \quad \sum^{N} |a(n)|^2(T+O(n)$$

gives

$$\int_T^{2T} |\mathbb{D}_{k,N}(1/2+it)|^2\, dt \quad \sum_{n<N} \frac{d_k(n)^2}{n}(T+O(n))$$

I+ well known that

$$\sum_{n<N} d_k(n)^2 \quad \frac{a_k}{\Gamma(k^2)} N \log^{k} N$$

and

$$\sum_{n<N}\frac{d_k(n)^2}{n} \sim \frac{a_k}{\Gamma(k^2+1}\log^{k^2} N$$

where

$$a_k \quad \prod\left(\left(1 \quad p\right)^{k^2}\sum \frac{d_k^2(p^r}{p^r}\right)$$

Thus for $N \ll T$ we deduce that

$$\int_T^{2T}|\mathbb{D}_{k,N}(1/2+it)|^2\, dt \quad \frac{a_k}{\Gamma(k^2+1)} T\log^{k^2} N$$

Using this with $M, N \ll T$ and $MN = (T/2\pi)^k$, we obtain the classical estimates for $I_1(T)$ and $I_2(T)$.

If $k \geq 3$, the condition $MN = (T/2\pi)^k$ forces at least one of M or N to be $\gg T$, so we need the mean value of a long Dirichlet polynomial This requires good estimates for the additive divisor sums

$$D_k(x,h) = \sum_{n \leq x} d_k(n) d_k(n+h) .$$

A precise formula for the main term of $D_k(x,h)$ can be conjectured by a heuristic application of the circle method, but it has not been proved We guess that

$$D_k(x,h) = m_k(x,h) + O(x^{1/2+\epsilon})$$

uniformly for $1 \leq h \leq x^{1/2}$, where $m_k(x,h)$ is a certain smooth function of x.

Using this in the mean value theorem for long Dirichlet polynomials, we obtain

Conjecture. *Let $N = (T/2\pi)^{1+\eta}$ with $0 \le \eta \le 1$. Then*

$$\int_T^{2T} |\mathbb{D}_{k,N}(1/2+it)|^2 \, dt \sim w_k(\eta) \frac{a_k}{\Gamma(k^2+1)} T L^{k^2}$$

where a_k is the product over primes defined previously, and

$$w_k(\eta) = (1+\eta)^{k^2} \left(1 - \sum_{n=0}^{k^2-1} \binom{k^2}{n+1} \gamma_k(n) \left(1 - (1+\eta)^{-(n+1)} \right) \right) ,$$

where

$$\gamma_k(n) = (-1)^n \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{k}{i} \binom{k}{j} \binom{n-1}{i-1, j-1, n-i-j+1}$$

for $n \ge 1$, and

$$\gamma_k(0) = k.$$

The conjecture restricts us to $N \ll T^2$ Thus, M and N in the formula

$$\int_T^{2T} |\zeta(1/2+it)|^{2k}\,dt$$

$$\int_T^{2T} |\mathbb{D}_{k,N}(1/2+it)|^2\,dt + \int_T^{2T} |\mathbb{D}_{k,M}(1/2+it)|^2\,dt.$$

must satisfy

$$M \ll T^2 \quad N \ll T^2 \quad \text{and} \quad MN = (T/2\pi)^k$$

Writing $N = (T/2\pi)^{1+\eta}$ and $M = (T/2\pi)^{k-1-\eta}$, we find that

$$\int_T^{2T} |\zeta(1/2+it)|^{2k}\,dt \sim \int_T^{2T} |\mathbb{D}_{k,(T/2\pi)^{1+\eta}}(1/2+it)|^2\,dt$$
$$+ \int_T^{2T} |\mathbb{D}_{k,(T/2\pi)^{k-1-\eta}}(1/2+it)|^2\,dt\,,$$

where $0 \leq \eta \leq 1$.

Thus,

$$\int_T^{2T} |\zeta(1/2+it)|^{2k}\,dt$$
$$\sim \big(w_k(\eta) + w_k(k-2-\eta)\big)\frac{a_k}{\Gamma(k^2+1)}TL^{k^2}$$

Example: The 6th moment. Take $k = 3$. Then

$$\int_T^{2T} |\zeta(1/2+it)|^6 \, dt$$

$$\sim \big(w_3(\eta) + w_3(1-\eta)\big)\frac{a_3}{\Gamma(10)} T L^9$$

for $0 \le \eta \le 1$. We find from the conjecture that

$$w_3(\eta) = 1+9\eta+36\eta^2+84\eta^3+126\eta^4-630\eta^5+588\eta^6+180\eta^7-9\eta^8+2\eta^9\,,$$

and one can verify that

$$w_3(\eta) + w_3(1-\eta) = 42$$

for $0 \le \eta \le 1$. Therefore

$$\int_T^{2T} |\zeta(1/2+it)|^6 \, dt \sim 42\,\frac{a_3}{9!} T L^9$$

Example: The 8th moment. Take $k = 4$. Then

$$\int_T^{2T} |\zeta(1/2+it)|^8 \, dt$$

$$\sim \big(w_4(\eta) + w_4(2-\eta)\big)\frac{a_4}{\Gamma(17)} T L^{16}$$

where η and $2-\eta$ must be in $[0,1]$ This forces $\eta = 1$. But

$$w_4(1) = 12012\,.$$

Hence

$$\int_T^{2T} |\zeta(1/2+it)|^8 \, dt \sim 24024 \frac{a_4}{16!} T L^{16}$$

Originally we expected to be able to take $N > T^2$ in our formulas. That means we expected the error terms in

$$D_k(x,h) = m_k(x,h) + O_h(x^{1/2+\epsilon})$$

to cancel in our long mean value theorems when we average over h up to $x^{1-\epsilon}$ instead of just up to $x^{1/2-\epsilon}$ We were surprised to find, however, that this is not the case.