Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Statistics of low-lying zeros of L-function and random matrix theory IV

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7 April 2004

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Selberg Class $\mathcal{S}$

- $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

* Analyticity: L is meromorphic in $\mathbb{C}$ with at most a finite number of poles on $\sigma = 1$

* Ramanujan:
$\forall \delta > 0, \exists c(\delta)$ st
$$|a(n)| \leq c(\delta) n^{\delta}$$

* Functional Equation
$\exists Q > 0, w_j > 0, \mu_j : \mathrm{Re}\, \mu_j \geq 0$ st ϵ in $\mathbb{C}$
$$Q^s \prod_{j=1}^{J} \Gamma(w_j s + \mu_j) L(s) =: \Lambda(s) = \epsilon \overline{\Lambda(1-\bar{s})}$$

* Euler Product
$$L(s) = \prod_p L_p(p^{-s}), \quad L_p(p^{-s}) = \exp\left(\sum \frac{b_{n,p}}{n^s}\right)$$
$b_{n,p} = 0$ unless n is a power of p

$\exists \theta < 1/2 : b_{n,p} \ll n^{\theta}$

$d = 2 \sum_{i=1}^{J} w_j$ is the degree

$N(T) \sim d \frac{T}{\pi} \log T$

L is prim if $L \neq L_1 L_2$

- H_k = set of wt. k cusp forms with mult. coeffs (for $SL_2(\mathbb{Z})$)

- $\# H_k = \begin{cases} \left[\frac{k}{12}\right] & \text{if } k \not\equiv 2 \bmod 12 \\ \left[\frac{k}{12}\right] - 1 & \text{if } k \equiv 2 \bmod 12 \end{cases}$

$f \in H_k$

- $L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \quad \sigma > 1$

(Deligne) $|\lambda_f(n)| \le d(n)$

- $\lambda_f(n)\lambda_f(m) = \sum_{d \mid (m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$

- $\Lambda(s) = (2\pi)^{-s}\Gamma\left(s + \frac{k-1}{2}\right)L_f(s) = (-1)^{\frac{k}{2}}\Lambda(1-s)$

Symm square L fcn

$$L(s, \mathrm{sym}^2(f)) \quad \zeta(2s)\sum_{n=}^{\infty} \frac{\lambda_f(n^2)}{n^s}$$

$$\Lambda(s, \mathrm{sym}^2 f) = \pi^{\frac{3s}{2}} \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right) L(s, \mathrm{sym}^2 f)$$

satisfes

$$\Lambda(s, \mathrm{sym}^2 f) = \Lambda(1 \; s, \mathrm{sym}^2 f)$$

$$L(s, \mathrm{sym}^2 f) \quad \prod_p \left(1 \;\; \frac{\alpha_f(p)^2}{p^s}\right)^{-} \cdot \left(1 \;\; \frac{\alpha_f(p)\beta_f(p)}{p^s}\right) \left(1 \;\; \frac{\beta_f(p)^2}{p^s}\right)$$

Maass forms

$$\Gamma_0(q) = \left\{ \underset{\substack{\shortparallel \\ \gamma}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} : \begin{array}{ll} ad-bc=1 & \\ q \mid c, & a,b,c,d \in \mathbb{Z} \end{array} \right\}$$

$$\gamma z = \frac{az+b}{cz+d}$$

Suppose that

$$f : \mathbb{H} \to \mathbb{C}$$

- $f(\gamma z) = f(z)$ for $\gamma \in \Gamma_0(q)$
- $\Delta f = -\lambda f$ $\quad \lambda = s(1-s)$, $s = \frac{1}{2} + it$

  where $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$
- $\displaystyle\int_{\Gamma \backslash \mathbb{H}} |f(z)|^2 \frac{dx\,dy}{y^2} < \infty$

$$f(z) = \sqrt{y} \sum_n \rho_\lambda(n) K_{it}(2\pi |n| y) e(nx)$$

Hecke operators:

$$T_n(f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right)$$

$$T_n(f) = \lambda_n f$$

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_n / n^s \qquad \left(\frac{q}{\pi}\right)^s \Gamma\left(\frac{s+it}{2}\right) \Gamma\left(\frac{s-it}{2}\right) L_f(s) = \pm \ldots (1-s).$$

Even + Odd

The L_f with $f \in H_k$ and $k \equiv 0 \mod 4$ form an even orthogonal family.

The L_f with $f \in H_k$ and $k \equiv 2 \mod 4$ form an odd orthogonal family.

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^{\frac{k}{2}} \cdots (k-s)$$

$k \equiv 0 \mod 4$ $+$

$k \equiv 2 \mod 4$ $-$

Orthogonality relations

- $\frac{1}{T}\int_0^T \left(\frac{m}{n}\right)^{it} dt \to \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$

- $$\sum_{\chi \bmod q}^{*} \chi(m)\bar{\chi}(n) = \sum_{\substack{d \mid m-n \\ d \mid q}} \mu(d)\varphi\left(\frac{q}{d}\right)$$

- $$\sum_{d\le x} \chi_d(n) \sim \begin{cases} \sum_{d\le x} 1 & \text{if } n=\square \\ \text{small} & \text{if } n\neq\square \end{cases}$$

- $$\sum_{f\in H_k(q)} \omega_f \lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^k \sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

- Kuznetsov formula

$$\sum_j h(t_j)\bar{\rho}_j(m)\rho_j(n) + \sum_{\mathfrak{a}} \frac{1}{4\pi}\int h(r)\bar{\varphi}_{\mathfrak{a}}(m,\tfrac{1}{2}+ir)\cdot \varphi_{\mathfrak{a}}(n,\tfrac{1}{2}+ir)\,dr$$

$$= \delta(m,n)H + \sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c} H^{+}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

$$H^{+}(x) = 2i\int_{-\infty}^{\infty} J_{2it}\, h(t)\, t\, (\cosh \pi t)^{-1}\, dt$$

H_k = set
space of cusp forms of wt k for $SL(2,\mathbb{Z})$ which are Hecke eigenfunctions (i.e. have multiplicative coeffs.)

- $f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz)$

Petersson Formula

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{\lambda_f(m)\lambda_f(n)}{L(\mathrm{sym}^2(f), 1)} = \delta(m,n)$$

$$+ 2\pi i^k \sum_{c=1}^{\infty} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

Here $S(m,n,c)$ is the Kloosterman sum: $S(m,n,c) = \sum_{\substack{x \bmod c \\ (x,c)=1}} e\left(\frac{mx + n\bar{x}}{c}\right)$

$(x\bar{x} \equiv \bmod c)$ J_{k-1} is the Bessel fcn

Families

Unitary

$\{\zeta(\tfrac{1}{2}+it) \quad t\in\mathbb{R}\}$

$\{L(s,\chi) : \chi \text{ prim mod } q\}$

$\{L(\tfrac{1}{2}+it) : t\in\mathbb{R}\}$

$\{L_f(\tfrac{1}{2},\chi) \quad \chi \text{ prim mod } q\}$

Orthog

$\{L_f(1/2) : f\in H_k(\ )\}$

$\{L_f(1/2) \quad f\in H_2(q)\}$

$\{L_f(1/2,\chi_d) \quad \chi_d \text{ real + prim}\}$

$\{L_{t_j}(1/2) : \text{Maass forms of eigenvalue } t_j\}$

$\{L_E(1/2) : \text{family of elliptic curves}\}$

Symp:

$\{L(1/2,\chi_d) : \chi_d \text{ real prim}\}$

$\{L(1/2, \operatorname{sym}^2 f) \quad f\in H_k(1)\}$

$\{L(1/2, \operatorname{sym}^2 f) : f\in H_2(q)\}$

1 Leve Density

Recall that the 1-level density functions are

$W(U)(x) \quad 1$

$W(SO^+)(x) = 1 + \frac{\sin 2\pi x}{2\pi x}$

$W(SO^-)(x) = \delta(x) + \frac{\sin 2\pi x}{2\pi x}$

$W(USp)(x) = - \frac{\sin 2\pi x}{2\pi x}$

$W(O)(x) \quad 1 + \frac{1}{2}\delta(x)$

Fourier transforms of 1-level density

$\delta(x)$	U
$\frac{1}{2}\chi_{[0,1]}(x) + \delta(x)$	SO^{+}
$\frac{1}{2}\chi_{[0,1]}(x) + \delta(x) + 1$	SO
$\frac{1}{2}\chi_{[0,1]}(x) + \delta(x)$	USp

Note The last 3 al have a jump d scontinuity at 1

Density Conjecture

Let $\mathcal{F}$ be a family, ordered by conductor $c(f)$

L.f. zeros
RH are $\frac{1}{2} + i\gamma_f$

Then

$$\lim_{X\to\infty} \frac{\displaystyle\sum_{\substack{f\in\mathcal{F}\\ c(f)\le X}} \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log c(f)}{2\pi}\right)}{\displaystyle\sum_{\substack{f\in\mathcal{F}\\ c(f)\le X}}}$$

$$= \int_{-\infty}^{\infty} \phi(x)\, W(\mathcal{F})(x)\, dx$$

where

$$W(\mathcal{F})(x) = \begin{cases} W(U)(x) \\ W(SO^+)(x) \\ W(SO^-)(x) \\ W(O)(x) \\ W(USp)(x) \end{cases}$$

depends on the symmetry type of the family

1 leve den ty resu ts

	Family	Supp of ϕ
Ozluk Snyder	$\{\chi_d \text{ real}\}$ ymp	$[2, 2]$
Hughes-Rudnick	$\{\chi \bmod q\}$ un tary	$[2 \quad 2]$
Iwaniec Luo Sarnak	$\{f \in H_k^+ \}$ L_f	$[\quad 2]$
	$\{f \in H_2^{\pm}(q)\}$ $SO^{\pm}$ L_f	
	$\{f \in H_k\}$ symp $L(\mathrm{sym}^2 f)$	$[\frac{7}{3}, \frac{7}{3}]$ with add. Hyp
Young	var ous fa es of ell t c curves	
Rub nste n	n level dens ties	$[- \quad 1]$

1- level densities for modular form L-fcns.

Consider

$$A_+(K,\phi) = \sum_{k \equiv 0 \bmod 4} h\left(\frac{k-1}{K}\right) \frac{2\pi^2}{k-1} \cdot$$

$$\sum_{f \in H_k} \frac{1}{L(1, \mathrm{sym}^2 f)} \sum_{\gamma_f} \phi\left(\frac{\gamma_f \log K^2}{2\pi}\right)$$

for suitable test functions h, ϕ

Here the zeros of L_f are $\frac{1}{2} + i\gamma_f$

The sum over k is concentrated around K. For each k there are $\sim \frac{k}{12}$ cusp forms f. Each f has about $\frac{\log k^2}{2\pi}$ zeros per unit interval.

Hughes - Rudnick

$$\lim_{\substack{q \to \infty \\ q \text{ prime}}} \frac{1}{q\ 2} \sum_{\substack{\chi \neq \chi_0 \\ \bmod q}} \sum_{\gamma_\chi} \phi\left(\frac{\gamma_\chi \log q}{2\pi} \right)$$

$$\int_{-\infty}^{\infty} \phi(x)\, dx$$

provided that

$$\operatorname{supp} \phi \subset [2, 2]$$

* Does not assume GRH

Rubinstein verified

n-level density for several families of modular form L-functions and for test functions ϕ with

$\operatorname{supp} \phi \subset [-,]$

Ozluk - Snyder

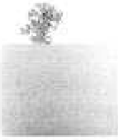
$\boxed{\text{GRH}}$

If $\operatorname{supp} \phi \subset [-2, 2]$ then

$$\lim_{D \to \infty} \frac{1}{2D} \sum_{d} e^{-\pi d^2/D^2} \sum_{\gamma_d} r\left(\frac{\gamma_d \log D}{2\pi}\right)$$

$$\int_{-\infty}^{\infty} \left(1 \quad \frac{\sin 2\pi x}{2\pi x} \right) r(x)\, dx$$

Here $\frac{1}{2} + i\gamma_d$ are the zeros of $L(s, \chi_d)$ for real quadratic characters χ_d



Theorem: (Iwaniec, Luo, Sarnak). Assume GRH. Then for ϕ with $\mathrm{supp}(\hat\phi) \subset [-2,2]$

$$\lim_{K\to\infty} \frac{A_+(K,\phi)}{\sum_{k \equiv 0 \bmod 4} h\left(\frac{k-1}{K}\right) \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{1}{L(1,\mathrm{sym}^2 f)}}$$

$$= \int_{\mathbb{R}} \phi(x)\, W(SO^+)(x)\, dx$$

and

$$\lim_{K\to\infty} \frac{A_-(K,\phi)}{\sum_{k \equiv 2 \bmod 4} h\left(\frac{k-1}{K}\right) \frac{2\pi^2}{k-1} \sum_{f} \frac{1}{L(1,\mathrm{sym}^2 f)}} = \int_{\mathbb{R}} \phi(x)\, W(SO^-)(x)\, dx$$

Thm (cont'd)

For ϕ with $\operatorname{supp}\phi \subset \left[-\frac{4}{3}, \frac{4}{3}\right]$

$$\lim_{K\to\infty} \frac{B(K,\phi)}{\displaystyle\sum_{\substack{k\equiv 0 \bmod 2 \\ k\le K}} \# H_k} \quad \int_{\mathbb{R}} \phi(x)\, W(USp)(x)\,dx$$

Matthew Young

Familes of ell ptc curves (show orthogonal statistics)

$\mathcal{F}: y^2 = x^3 + ax + b$ $\quad a \approx A = X^{1/3}$, $b \approx B = X^{1/2}$

$\operatorname{supp} \phi \subset \left[-\frac{7}{9}, \frac{7}{9}\right)$

$\mathcal{F}_2: y^2 = x^3 + ax + b^2$ $\quad a \approx A = X^{1/3}$, $b \approx B = X^{1/4}$

$\operatorname{supp} \phi \subset \left(\frac{23}{48}, \frac{23}{48} \right)$

Thm (GRH)

$$\lim_{X\to\infty} \frac{\sum_{f \in \mathcal{F}_1} w(f) \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log X\right)}{\sum_{f \in \mathcal{F}} w(f)} = \int_{-\infty}^{\infty} \left(1 + \frac{1}{2}\delta(x)\right) \phi(x)\, dx$$

$$\left(\text{Same for } \mathcal{F}_2\right) = \int_{-\infty}^{\infty} \left(1 + \frac{3}{2}\delta(x)\right) \phi(x)\, dx$$

Explicit Formula

Suppose that $L(s) \quad \sum \frac{\lambda_n}{n^s}$ is entire

and satisfies $L(s) = X(s) L(1-s)$

Let $\frac{L'}{L}(s) - \sum_{n=1}^{\infty} \frac{-\Lambda_L(n}{n^s}$

Let $F(s)$ be a suitable test function and let $\phi(t) = F(\frac{1}{2}+it)$

be Schwarz and $\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e(-xt)\,dt$

be of compact support

Then, denoting zeros of L by $\frac{1}{2}+i\gamma$,

$$\sum_{\gamma} \phi(\gamma) \qquad 2\pi \int_{\infty}^{\infty} \frac{X}{X}\left(\tfrac{1}{2}+it\right)\phi(t)\,dt$$

$$\sum_{n} \frac{\Lambda_L(n}{\sqrt{n}} \left(\hat{\phi}\left(\frac{\log n}{2\pi}\right) + \hat{\phi}\left(-\frac{\log n}{2\pi}\right)\right)$$

Idea of Proof

$$2\pi i \int_{2} \frac{L}{L}(s) F(s)\,ds$$

$$- \sum_{n} \Lambda_L(n \quad \perp \int F(s) n^{s}\,ds$$

$$- \sum_{n} \frac{\Lambda_L(n)}{\sqrt{n}} \hat{\phi} \frac{\log n}{2\pi})$$

in the tegral, $\frac{L}{L}(s) - \frac{X'}{X}(s) \quad \frac{L}{L}(1-s)$ move path change ubles etc

$$X(s) = \frac{Q^{-s} \prod_{j=1}^{d} \Gamma\left(\frac{1-s}{2} + \overline{\mu_j}\right)}{Q^{s} \prod_{j=1}^{d} \Gamma\left(\frac{s}{2} + \mu_j\right)}$$

$$\frac{X'}{X}(s) = 2\log Q + \frac{1}{2} \sum \left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\ \frac{1-s}{2} + \overline{\mu_j}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{1-s}{2} + \mu_j\right) \right)$$

$$\frac{X'}{X}\left(\frac{1}{2} + t\right) = 2\log Q + \mathrm{Re} \sum_{j=1}^{d} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{t}{2} + \mu_j\right)$$

Let $\phi(x) = f\left(x \frac{\log R}{2\pi}\right)$ Assume $f(x) = f(-x)$

Exp c't formula

$$\sum_{\gamma} \hat{\phi}\left(\frac{\gamma \log R}{2\pi}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(2\log Q + \mathrm{Re} \sum_{j=1}^{d} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{t}{2} + \mu_j\right)\right) f\left(\frac{t \log R}{2\pi}\right) dt$$

$$- \frac{4\pi}{\log R} \sum \frac{\Lambda_L(n)}{\sqrt{n}} \hat{f}\left(\frac{\log n}{\log R}\right)$$

$$\frac{\log Q^2}{\log R} \hat{f}(0)$$

$\mathrm{supp}\, f \subset [a, a]$

$\Rightarrow n \leq R^{a}$

need only consider $n = p$ and $n = p^2$

Coeff. L

$$L(s) = \prod \prod_{j=1}^{d} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}$$

$|\alpha_j(p)| = 1$ or 0

$$\frac{L'}{L}(s) = \sum_p \sum_j \frac{d}{ds}\left(-\log\left(1 - \frac{\alpha_j}{p^s}\right)\right)$$

$$= \sum_p \sum_j \frac{-\left(\frac{-\alpha_j}{p^s}\right)(-\log p)}{-\frac{\alpha_j}{p^s}}$$

$$= -\sum \log p \sum_j \frac{\frac{\alpha_j}{p^s}}{1 - \frac{\alpha_j}{p^s}}$$

$$= -\sum_p \log p \sum_{j=1}^{d} \sum_{m=1}^{\infty} \frac{\alpha_j^m}{p^{ms}}$$

$$= -\sum_p \log p \sum_{m=1}^{\infty} \frac{1}{p^{ms}} \left(\sum_{j=1}^{d} \alpha_j(p)^m\right)$$

a_{p^2} = p^{th} coeff of $L(\mathrm{sym}^2 f)$

$$\Lambda_L(p^m) = -\log p \sum_{j=1}^{d} \alpha_j(p)^m$$

$\Lambda_L(p) = -a_p \log p$, $\Lambda_L(p^2) = -(2a_{p^2} - (a_p)^2)\log p$

/ L(coefts con d)

$$L(s) = \prod_p \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{s}} + \quad \right)$$

$$\prod_p \prod_{=1}^{d} \left(- \frac{\alpha_j(p)}{p^s} \right)$$

$$\prod_p \left(+ \frac{\alpha}{p^s} + \frac{\alpha_1^2}{p} + \right)\left(+ \frac{\alpha_2}{p^s} + \frac{\alpha^2}{p^{2s}} + \right) \quad \left(1 + \frac{\alpha_d}{p^s} + \right)$$

- $a_p - \sum_{j=}^{d} \alpha_j(p)$
- $a_{p^2} = \sum_{j=}^{d} \alpha(p)^2 + \sum_{i<j} \alpha \quad \alpha_j = \sum_{\leq j} \alpha \alpha_j$

$$L(s, \mathrm{sym}^2) = \prod_p \prod_{\leq \leq j \leq d} \left(\quad \frac{\alpha_i \alpha_j}{p^s} \right)$$

$$= \prod_p \prod_{i \leq j} \left(+ \frac{\alpha \alpha}{p^s} + \frac{\alpha_i^2 \alpha^2}{p^{2s}} + \right)$$

$$a(\mathrm{sym}^2, p) = \sum_{i \leq j} \alpha \alpha_j \qquad a_{p^2}$$

$$\sum_{j}^{d} \alpha_j^2 \quad \left(\sum_{j=}^{d} \alpha_j \right)^2 \quad 2 \sum_{i<j} \alpha_i \alpha_j$$