Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

A new model for the Rieman zeta function

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Let $u(x)$ be a C_c^∞ function of total mass 1 whose support is contained in the interval $[1, e]$, and let $v(t) = \int_t^\infty u(x)\,\mathrm{d}x$ and $U(z) = \int_0^\infty u(x) E_1(z \log x)\,\mathrm{d}x$ where E_1 is the exponential integral.

Theorem. If $|t| \geq 2$ is not an ordinate of a zero of the zeta function, then for $2 \leq X \leq t^2$, we have for any positive integer K,

$$\zeta(\tfrac{1}{2} + \mathrm{i}t) = P(t;X) Z(t;X) \left(1 + \mathcal{O}(\frac{\sqrt{X}}{(|t| \log X)^K})\right)$$

where

$$P(t;X) = \exp\left(\sum_{n=2}^{\infty} \frac{\Lambda(n) n^{-\mathrm{i}t}}{\sqrt{n} \log n} v(e^{\log n / \log X})\right)$$

and

$$Z(t;X) = \exp\left(-\sum_{\gamma_n} U(\mathrm{i}(t - \gamma_n) \log X)\right)$$

The theorem follows from the following explicit formula, due to Bombieri and Hejhal.

Then, for $\sigma \geq 0$, $|t| \geq 2$, $2 \leq X \leq t^2$, with t not the ordinate of a zero of the zeta function,

$$
\begin{aligned}
-\frac{\zeta'}{\zeta}(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} v(e^{\log n / \log X}) \\
- \sum_{\rho} \frac{\widetilde{u}(1-(s-\rho)\log X)}{s-\rho} \\
+ \frac{\widetilde{u}(1-(s-1)\log X)}{s-1} \\
- \sum_{m=1}^{\infty} \frac{\widetilde{u}(1-(s+2m)\log X)}{s+2m}
\end{aligned}
$$

where

$$\widetilde{u}(s) := \int_0^{\infty} u(x) x^{s-1} \mathrm{d}x$$

Splitting conjecture:

$$\frac{1}{T}\int_T^{2T} \left|\zeta(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k} \mathrm{d}t$$

$$\sim \frac{1}{T}\int_T^{2T} |P(t;X)|^{2k}\,\mathrm{d}t \times \frac{1}{T}\int_T^{2T} |Z(t;X)|^{2k}\,\mathrm{d}t$$

Theorem. If $X = \mathcal{O}(\log T)$ and $u \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} u \subseteq [e^{1-\epsilon}, e]$, then for fixed real $k > 0$, we have

$$\frac{1}{T}\int_T^{2T} |P(t;X)|^{2k}\,\mathrm{d}t \sim a(k) e^{\gamma k^2} (\log X)^{k^2}$$

Conjecture. If $X, T \to \infty$ such that $\frac{\log T}{\log X} \to \infty$, then for fixed k subject to $\mathfrak{Re}(k) > -1/2$,

$$\frac{1}{T}\int_0^T |Z(t;X)|^{2k}\,\mathrm{d}t$$

$$\sim \frac{G^2(k+1)}{G(2k+1)} \left(\frac{\log T}{\log X}\right)^{k^2} e^{-\gamma k^2}$$

Sketch of proof of theorem. Set

$$P(t;X)^k = \prod_p \left(\sum_{m=0}^{\infty} \frac{a_k(p^m)}{p^{m/2} p^{imt}} \right)$$

Note that since $v(t) = \int_t^\infty u(x)\,\mathrm{d}x$, we have

$$v(t) = \begin{cases} 1 & \text{if } t \le e^{1-\epsilon} \\ 0 & \text{if } t \ge e \end{cases}$$

and so we have the following estimates:

- $0 \le a_k(n) \le d_k(n)$ for all n
- $a_k(p^m) = d_k(p^m)$ if $m \le \left\lfloor (1-\epsilon)\frac{\log X}{\log p} \right\rfloor$
- $a_k(p^m) = 0$ for all $m \ge 1$, if $p > X$.

The mean value theorem of Montgomery and Vaughan gives

$$\frac{1}{T}\int_T^{2T} |P(t;X)|^{2k}\,\mathrm{d}t = \sum_{n=1}^{\infty}\frac{a_k^2(n)}{n} + \mathcal{O}\left(\frac{1}{T}\exp\left(\frac{X}{\log X}\right)\right)$$

Using the estimates for $a_k(p^m)$, we can show that if $\operatorname{supp} u \subseteq [e^{1-\epsilon}, e]$, with $\epsilon > 0$ arbitrarily small, then as $X \to \infty$,

$$\prod_{p\le X}\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\frac{a_k^2(p^m)}{p^m} = a(k)\left\{1+\mathcal{O}(\epsilon)+\mathcal{O}(\frac{1}{\log X})\right\}$$

where

$$a(k) = \prod_p\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}.$$

Since

$$\prod_{p \le X} \left(1 - \frac{1}{p}\right)^{-k^2} = e^{\gamma k^2} (\log X)^{k^2} \left(1 + \mathcal{O}_k(\frac{1}{\log X})\right)$$

if we choose $\epsilon = 1/\log X$, we have

$$\frac{1}{T} \int_T^{2T} |P(t;X)|^{2k} \, \mathrm{d}t = a(k) e^{\gamma k^2} (\log X)^{k^2} + \mathcal{O}\left(\frac{1}{\log X}\right) + \mathcal{O}\left(\frac{\exp(X/\log X)}{T}\right)$$

with the error terms being small, as $X = \mathcal{O}(\log T)$.

Recall that

$$Z(t;X) = \exp\left(-\sum_{\gamma_n} U(\mathrm{i}(t-\gamma_n)\log X)\right)$$

where

$$U(z) = \int_0^\infty u(x)\Bigg(-\log(z\log x) - \gamma + \int_0^{z\log x} \frac{1-e^{-y}}{y}\,\mathrm{d}y\Bigg)\,\mathrm{d}x$$

Thus if $\operatorname{supp} u \in [e^{1-\epsilon}, e]$, then $|Z(t;X)|$ acts like a characteristic polynomial,

$$|Z(t;X)|^{2k} \approx \prod_{|t-\gamma_n|\ll 1/\log X} |(t-\gamma_n)e^{\gamma}\log X|^{2k}$$

Not quite a local statistic, but one might still believe the random matrix conjecture that

$$\frac{1}{T}\int_0^T |Z(t;X)|^{2k}\,\mathrm{d}t$$

$$\approx \left\langle \prod_{|T-\gamma_n|\ll 1/\log X} |T-\gamma_n e^{\gamma}\log X|^{2k} \right\rangle$$

$$\sim \frac{G^2(k+1)}{G(2k+1)}\left(\frac{\log T}{e^{\gamma}\log X}\right)^{k^2}$$

Hence

$$\frac{1}{T}\int_T^{2T} \left|\zeta(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k}\mathrm{d}t \sim$$

$$a(k)e^{\gamma k^2}(\log X)^{k^2} \times \frac{G^2(k+1)}{G(2k+1)}\left(\frac{\log T}{e^{\gamma}\log X}\right)^{k^2}$$

$$= a(k)\frac{G^2(k+1)}{G(2k+1)}(\log T)^{k^2}$$