Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

Conjectures for moments of a family of L-functions

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Conjectures for moments of a family of L functions

Joint work with

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preprint to be available soon ($\le$ May 1)

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Functional equation for characteristic polynomials

$$\Lambda_A(x) - \det(I \quad xA) \qquad A \; N\times N \text{ unitary}$$

$$(\;\; {}^N \det A \, x^N \det{}^{/} \;\; {}_x A \;\; + I)$$

$$(\;\; {}^N \det A \, x^N \det(I \quad {}_x A^t)$$

$$(\;\; {}^N \det A \, x^N \Lambda_{\bar{A}}\left(\tfrac{1}{x}\right)$$

$$L(s \qquad \varepsilon \qquad \chi(s \quad L(-s)$$

$A \in SO(2N) \qquad (-)^{2N} \det A$

$A \in SO(2N+ \qquad (-^{2N+1} \det A \;\; -$

$A \in Sp\,2N \qquad (-^{N} \det A$

function

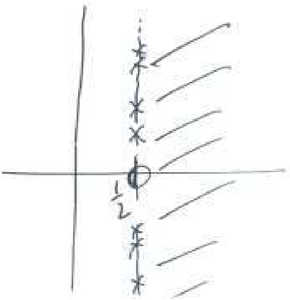
$\Sigma a_n n^s$

$s = \Sigma\ X(s \quad L \quad s$

$\Sigma =$

$X(\frac{1}{2} =$

$|X\ \frac{1}{2} + t)$



$\lim\limits_{s \to +\infty} L \quad =$

$X(\frac{1}{2}$

$L(\frac{1}{2} + x$

RMT

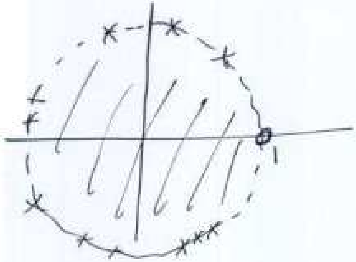
$\Lambda_A(x) = \det(I - xA)$

$\Lambda_A(x) = (-1)^N \det A\ x^N \Lambda_{\bar{A}}(\frac{1}{x})$

$|(-1)^N \det A| = 1$

$(1)^N = 1$

$|(e^{i\theta})^N| = 1$



$\Lambda_A(0) = 1$

conductor" N

$\Lambda_A(e^{-x})$

The "real version" of L and Λ

By the functional equation,

$$Z_L(s) \quad \varepsilon^{1/2}\, \chi_L(\ )^{1/2} L(s)$$

s rea on the $\frac{1}{2}$ line (not standard)

and $Z_L(\frac{1}{2}+t \quad + \ |L(\frac{1}{2}+t)|$

<u>Exercise</u> Compute

$$\int_0^T Z_\zeta(\tfrac{1}{2}+t\ dt \quad \text{and} \quad \int_0^T |Z_\zeta(\tfrac{1}{2}+t)\ ^2 dt$$

Conclude that ζ s has infinitely many zeos on the critical line

Also think about the difference between

$$\int_0^T Z_\zeta(\tfrac{1}{2}+t\ dt \quad \text{and} \quad \int_0^T \zeta(\tfrac{1}{2}+t)\, dt$$

Similarly,

$$Z_A(s) \quad (-1)^{N/2}(\det A\ ^{1/2}\, s^{\ N/2}\, \Lambda_A(s$$

s real on the unt circle

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A family of L functions $\{L_f \quad f \in \mathcal{F}\}$

L_f is a primitive L function

$\Lambda_f(s \quad Q_f^s \prod \Gamma(\omega_{j,f} s + \mu_{j,f}) L_f(s$

$\varepsilon_f \Lambda_f(1-s)$

If $\mathcal{F}$ is finite, then $Q_f, \omega_{j,f}, \mu_{j,f}$ are a independent of f

If $\mathcal{F}$ is infinite then $Q_f, \omega_{j,f}, \mu_{j,f}$ are monotonic functions of conductor(f)

ε_f either constant

or $\varepsilon_f \pm 1$ and $\langle \varepsilon_f \rangle \; 0$

or ε_f is famly distributed o the unit circle

$L_f \quad L \otimes \mathcal{L}_f$ Rankin Se eg convolution

L and $\mathcal{L}_f$ are primitive L function

$\sum \frac{a}{n^s} \otimes \sum \frac{b_n}{n^s} \quad \sum \frac{a \, b_n}{n^s}$ if at leat one of them ha degree 1

$$h_f(s \quad \sum_{n}^{\infty} \frac{f(n}{n^s}$$

such that

$$\langle f(n) \quad f(n_l)\overline{f(n_{l+1})} \quad - \overline{f(n_k)} \rangle$$

$$\delta(n, \quad , n_k)$$

s a well defined multiplic tive function

$$\delta(m_1 n_1 \quad m_k n_k) - \delta \, m \quad , m_k \; \delta(n \quad n_k$$

$$f \; (m \quad m_k, n \quad n_k)$$

The point

$$\sum_{n_1 \quad n_k}^{\infty} \frac{\delta(\quad n_k}{n^{s_1} \quad n_k^{s_k}} \quad \prod_p \sum_{e_1 \quad e_k \; 0}^{\infty} \frac{\delta \; p^{e_1} \quad , p^{e_k})}{p^{e_1 s_1} \quad p_k^{e_k s_k}}$$

Need a formula for $\delta(p^{e_1}, \quad , p^{e_k})$

n order to make an explicit conjecture

Examples of $\delta(n \quad n_k)$ $\quad 5\frac{1}{2}$

$$L_y(s \quad \zeta(s+y) \quad \sum_n^\infty \frac{n^{-\iota}}{n^s} \quad \sum_n^\infty \frac{y(n}{n^s}$$

$$y\langle \quad y\, n_1)\, y(n_{l+}) \quad y(n_k \rangle$$

$$\langle n^{\,y} \quad n_l^{\,\iota}\; n_l \quad n_k \rangle$$

$$\left\langle \left(\frac{n_{l+1} \quad n_k}{n \quad n_l} \right)^{\iota y} \right\rangle$$

$$\begin{cases} & f \quad n \quad n_l - n_{l+} \quad n_k \\ 0 & \text{otherwise} \end{cases}$$

$- \ \delta(n \quad n_k)$ which mu tip cotive

$$L_{\chi_d}(s) - L(s\ \chi_d) \quad \sum \frac{\chi_d(n)}{n^s}$$

d a fundamenta discriminant

d a mod N

$$\langle \chi_d(n \quad \chi_d(n_k) \rangle$$

$$-\langle \chi_d(n_1 \quad n_k) \rangle \begin{cases} * & n \quad n_k \ g\square,\ g/\Lambda \\ 0 & \text{otherwise} \end{cases}$$

Question Is the set of L functions (5 3/4)

$\mathcal{F}_{A,B}$ $\{ L_{a,}$ (s L associated to the elliptic curve y^2 $x + ax+b$: $a \in A$, $b \in B$ $\}$

a family

The issues The conductor is not a simple function of a and b (probably ok)

- For reasonable A and B (eg $A - \mathbb{Z}$, B $\square$)

$\langle f($ $f(n) \rangle$ ex ts

obviously? s mu tiplicative?

I have no idea how one would compute t

Recipe   Conjecturing mean values

) Start with $L(s,\alpha \quad ,\alpha_k)$ $\mathcal{L}(s+\alpha)$ $\mathcal{L}(s+\alpha_k$

2 Replace each $\mathcal{L}$ function by an appx. fcl eqn

$$\mathcal{L}(s) - \sum \frac{a_n}{n^s} + \varepsilon \chi(s) \sum \frac{a_n}{n^s} + \underset{\text{ignore}}{\text{remainder}}$$

3) Multiply out to obtain 2^k terms of the for

$$(\ \text{roduct of } \varepsilon)(\text{product of } \chi) \sum_{n_1 \quad n_k} (S \quad \text{mmand})$$

4) Replace each (roduct of ε) by its expected value

5 Replace each (summand) by ts expected vaiu

6) Com plete the resu ting sums, ca the

ts $M(\ ,\alpha \quad ,\alpha_k)$

Conjecture

$$\langle L(\ \alpha \quad ,\alpha_k) \rangle \quad \langle M(s,\alpha, \quad \alpha_k) \rangle$$

7) Express the answer in a nice forn

The answer tells you the symmetry type

Conjecture / Theorem   U itary case

$$(\quad \begin{cases} Z_L(\tfrac{1}{2}+x) & \text{p imitive } L \text{ function } L(\ + t \\ Z_A(e^{\ x}) & A \in U(N \end{cases}$$

$$G(w, \ , w_{2k}) - \begin{cases} A_k(w, \ , w_{2k}) \prod_{i,j}^{k} \zeta(1+ \ _i \ -_{k+j}), & A_k(0 \ ) \ a_k \\ \prod_{i,j}^{k} (1-e^{\ z \ + z_{j+k}}) \end{cases}$$

$$C \text{ Conductor of } Z \quad \begin{cases} w \log\left(\frac{Q^{2/w} t}{2}\right) & \text{e.g. } \log\left(\frac{t}{2\pi}\right) \text{ for the } \zeta\text{-function} \\ N \end{cases}$$

$$\langle Z(\alpha \qquad Z(\alpha_{\ k})\rangle$$

$$\frac{\ }{\ }^{k} \ 2\pi \ ^{2k} \oint \oint \frac{G(w \qquad , w_{2k}) \Delta^2(w \qquad w_{\ k}}{\prod_{i}^{2k} (w \quad \alpha \ )} e^{\frac{C}{2} \sum^{k} w \quad w_{k+}} dw_1 \ dw_{2k}$$

$$7(\alpha \quad 7(\alpha_n)) \begin{cases} \dfrac{\int_{\infty} Z_L(\tfrac{1}{2}+\alpha \quad Z \quad 1+\alpha_n \; g(t)\,dt}{\int_{\infty}^{\infty} g(t)\,dt}, & g \quad \lambda_{[0,T]} \text{ acceptable, and gives } O(T^{\text{loss}}) \text{ error term} \\ \int_{U(N)} Z_A\, e^{\alpha} \quad Z_A\, e^{-\alpha_n})\, dA_{\text{Haar}} \end{cases}$$

$$A_n(w \quad n \quad \prod_{p} \prod_{i,j}^{n} \Big(\quad \frac{}{p^{\,+w} \quad w_{n+}} \Big) \int_0 \prod^{n} h_p\Big(\frac{e(\theta}{p} \quad h_p\Big(\frac{e(\theta)}{p^{\,i-\alpha_{n+j}}}\Big) \, d\theta$$

where L s $\prod_p h_p(p^{s})$

Can let $\alpha \to 0$ to recover previous conjectures

Rec pe $\Longrightarrow$ the random matr x c njectures

Prehistoric vers on

<u>Conjecture</u> F 92]

$$T \int_0^T \frac{\zeta(\frac{1}{2}+a+it)\,\zeta(\quad +b \quad)}{\zeta(\ \ _2+c+\ t)\,\zeta(\frac{1}{2}+d+i\ )}\,dt \qquad +(\quad T^{a\ b})\frac{(a-c)(b-d)}{(a+b)(c+d)}$$

$$\text{Con} \Longrightarrow \quad T \int_0^T \Big|\frac{\zeta'}{\zeta}(\tfrac{1}{2}+a \quad it)\Big|^2 \qquad \frac{T}{4a} \qquad (1)$$

<u>Goldston Gonek Montgomery</u>

$$(2) \qquad T \int^T \Big|\frac{\zeta'}{\zeta}(\ \ _2+a+\ t)\Big|^2 dt \qquad \log^2 T \int^{\infty} F(\alpha,T)\,T \ d$$

(1)+(2) $\Longrightarrow$ Montgomery's pair correlation conjecture

The recipe actu lly g ves

$$\int_0^T \frac{\zeta(\ \ +a+it\ \ \zeta(\tfrac{1}{2}+b\ \ \ )}{\zeta(\tfrac12+c\ \ \ )\,\zeta(\tfrac12+d\ \ \ t} \, dt$$

$$\int_0^T YA(a\ b,c\ d\ \ + \left(\frac{t}{2\pi}\right)^{\ \ a\ b} YA(\ b,-a,c,d)\, dt$$

$+$ small error

Where

$$Y(a,b\ c,d \qquad \frac{\zeta(\ \ +a+b)\ \zeta(\ \ +c+d}{\zeta(\ \ +a+c\ \ \ \zeta(\ \ +b+d}$$

$$A(a,\ \ c,d \qquad \prod_p \frac{(\ \ \ \ p^{\ 1\ a\ b})(\ \ \ \ p^{\ 1\ c\ d}}{(1-p^{-1-a-d})(1-p^{-1-b-c})}$$

$$\times \int_0^1 \frac{\left(1\ \ \ \frac{e(\theta)}{p^{\frac12}\ \ c}\right)\left(\ \ \ \ \frac{e(\theta)}{p^{\frac12+d}}\right)}{\left(\ \ \ \ \frac{e(\theta}{p^{\frac12}\ \ a}\right)\left(\ \ \ \ \frac{e(\theta)}{p^{\frac12+b}}\right)} \, d\theta$$

Thi implies the old conjecture, and can also be used to obtain lower order terms in the pair correlation Th s sh ld agre with the wer order terms obtained by Bogolmony and Keating