Isaac Newton Institute for Mathematical Sciences
RMAw02 - Recent Perspectives in Random Matrix Theory and Number Theory

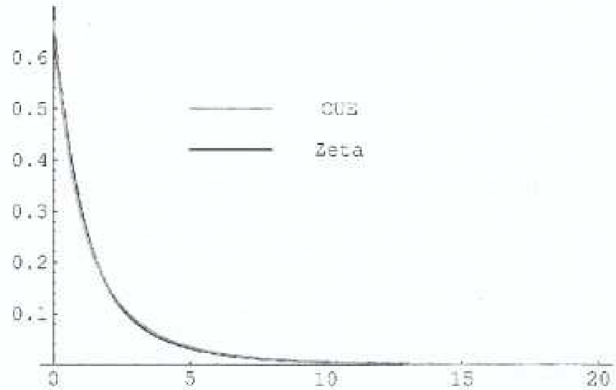
RMT moment calculations III

J.P. Keating (Bristol)

8 April 2004

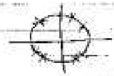
Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road, Cambridge CB3 0EH, UK

Tel: +44 1223 335999 Fax: +44 1223 330508
E-mail: webseminars@newton.cam.ac.uk
http://www.newton.cam.ac.uk/webseminars


0.6
0.5
0.4
0.3
0.2
0.1
0
5
10
15
20
CUE
Zeta

Cambridge Lectures (II)

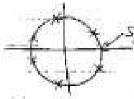
$A \in U(N)$



eigenvalues $e^{i\theta_n}$

$$Z(A,\theta) = \prod_{n=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right)$$

$A \in USp(2N)$, $O(N)$



eigenvalues $e^{\pm i\theta_n}$

$$Z(A,\theta) = \prod_{n=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right)\left(1 - e^{i(-\theta_n - \theta)}\right)$$

<u>Moments</u>

- Weyl integration formula
- Selberg integral

$$\int_{USp(2N)} Z(A,0)^s \, d\mu(A) = 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\,\Gamma(\frac{1}{2}+s+j)}{\Gamma(\frac{1}{2}+j)\,\Gamma(1+s+N+j)}$$

$$= M_{Sp}(s, N)$$

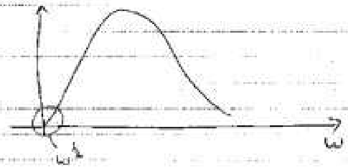
- $\log Z$ – central limit theorem

- $$\lim_{N\to\infty} \frac{1}{N^{s(s+1)/2}} \int_{USp(2N)} Z(A,0)^s \, d\mu(A)$$

$$= 2^{s^2/2} \frac{G(1+s)\sqrt{\Gamma(1+s)}}{\sqrt{G(1+2s)\,\Gamma(1+2s)}} \equiv f_{Sp}(s)$$

$$f_{Sp}(n) = \frac{1}{\prod_{j=1}^{n}(2j-1)!!} = \frac{1}{(2n-1)(2n-3)^2(2n-5)^3 \cdots}$$

$$\int_{A \in Sp(2N)} \delta(Z(A,0)-\omega)\, d\mu(A) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_{Sp}(s)\,\frac{ds}{\omega^{s+1}}$$



$$\int_{SO(2N)} Z(A,0)^s\, d\mu(A) = 2^{2Ns}\prod_{j=1}^{N}\frac{\Gamma(N+j-1)\,\Gamma(s+j-\frac{1}{2})}{\Gamma(j-\frac{1}{2})\,\Gamma(s+j+N-1)}$$

$$= M_O(s;N)$$

- $\log Z$ – central limit theorem

- $$\lim_{N\to\infty}\frac{1}{N^{s(s-1)/2}}\int_{SO(2N)} Z(A,0)^s\, d\mu(A)$$

$$= 2^{s/2}\,\frac{G(1+s)\sqrt{\Gamma(1+2s)}}{\sqrt{G(1+2s)\,\Gamma(1+s)}} \equiv f_O(s)$$

$$f_O(n) = 2^n f_{Sp}(n-1)$$

NB $\quad f_O(s)\, f_{Sp}(s) = 2^{s^2} f_U(s)$

$$\int_{SO(2N)} \delta(Z(A,0)-\omega)\, d\mu(A) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_O(s;N)\,\frac{1}{\omega^{s+1}}\, ds$$

Applications ......

Families of L-functions + Symmetry

eg 1 Dirichlet L-functions

$$\chi_d(p) = \left(\frac{d}{p}\right) = \begin{cases} +1 & \text{if } p \nmid d \text{ and } x^2 \equiv d \pmod p \text{ soluble} \\ 0 & \text{if } p \mid d \\ -1 & \text{if } p \nmid d \text{ and } x^2 \equiv d \pmod p \text{ not soluble} \end{cases}$$

$$L_D(s, \chi_d) = \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}$$

$$= \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$$

eg 2 L-functions associated with elliptic curves

$$f(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^2 (1 - e^{22\pi i n z})^2$$

$$= \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ s.t. } 11 \mid c$$

$$N_p = \# \{ (x,y) \in \mathbb{F}_p^2 : \underbrace{y^2 = 4x^3 - 4x^2 - 40x - 79}_{E_{11}} \}$$

then $\boxed{a_p = p - N_p}$

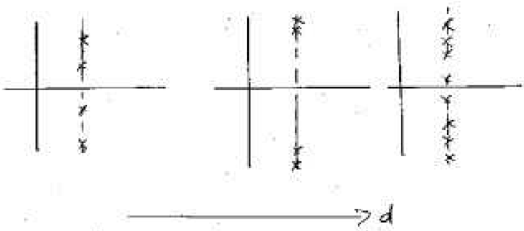
$$L_{E_{11}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$\boxed{L_{E_{11},d}(s) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^s}}$$

Functional equation

$$\Phi_{E_{11},d}(s) = \left(\frac{2\pi}{\sqrt{11}\,|d|}\right)^{-s}\Gamma(s)\,L_{E_{11},d}(s)$$
$$= \chi_d(-11)\,\Phi_{E_{11},d}(2-s)$$

$$\boxed{\text{take } \chi_d(-11) = +1}$$



Katz-Sarnak: eg1 - USp (2N)
eg2 - SO (2N)

Moments (Conrey-Farmer, K-Smith)

$$\frac{1}{X^{*}}\sum_{\substack{0<d<X\\ \chi_d(-11)=+1}} \Big(L\big(\underset{\text{eg1}}{\tfrac{1}{2}} \text{ or } \underset{\text{eg2}}{1},\ \chi_{-d}\big)\Big)^{s} \longleftrightarrow \int_{\substack{USp(2N)\ -\ \text{eg1}\\ SO(2N)\ -\ \text{eg2}}} (Z(A,0))^{s}\,dA$$

Formula for $L_{E_{11},d}(1)$ (Shimura, Waldspurger, Kohnen-Zagier)

for $d<0$, $\chi_d(-11)=+1$

$$\boxed{L_{E_{11},d}(1) = \frac{\kappa\, c_{|d|}^2}{\sqrt{|d|}}}$$

where

κ = constant $(= 2.91763\ldots)$

and

$$g(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}$$

satisfies

$$g\left(\frac{az+b}{cz+d}\right) = \varepsilon(a,b,c,d)\,(cz+d)^{3/2} g(z)$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ s.t. } 44 \mid c$$

$\boxed{\text{NB} \quad c_n \in \mathbb{Z}}$.

<u>Implications?</u> (Conrey, K, Rubinstein, Smith)

1. Generalization of the Sato-Tate Law

<u>Sato-Tate</u> Let $\frac{a_p}{\sqrt{p}} = 2\cos\theta_p \quad 0 \le \theta \le \pi$

(Hasse, Deligne)

Then

$$\lim_{X\to\infty} \frac{1}{\pi(X)} \#\{p<X : \alpha < \theta_p \le \beta\} = \frac{2}{\pi}\int_\alpha^\beta \sin^2\theta \, d\theta$$

where $\pi(x) = \#\{p < X\}$

<u>Generalization to half-integer weight modular forms?</u>

ie distribution of values of c_n

RMT $\Rightarrow$

a) $$\lim_{D\to\infty} \frac{1}{D^*}\left\{1<d\le D : \chi_d(-1)=1,\ \frac{2\log|c_{|d|}| - \frac{1}{2}\log d - \frac{1}{2}\log\log d}{\sqrt{\log\log d}} \in (\alpha,\beta)\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}\, dx$$

b) distribution for $c_{|d|}$ 

eg $$\lim_{\beta\to 0} \frac{\mathrm{Prob}(|c_{|d|}| < \beta)}{\beta} = \kappa' \, |d|^{-1/4} (\log|d|)^{3/8}$$

κ' 0.6456 . . .

2. Frequency of vanishing of L-functions

$$\text{if} \quad L_{E_{11},d}(1) < \frac{\kappa}{\sqrt{|d|}} \implies L_{E_{11},d}(1) = 0.$$

This implies:

Conjecture 1

$$\#\{p \le D : \chi_{-p}(-11) = 1,\ L_{E_{11},-p}(1) = 0\} \asymp \frac{D^{3/4}}{(\log D)^{5/8}}$$

Conjecture 2 (vanishing in arithmetic progressions)

Let
$$R_p(D) = \frac{\#\{d < D : \chi_d(-11) = 1,\ \chi_d(p) = 1,\ L_{E_{11},d}(1) = 0\}}{\#\{d < D : \chi_d(-11) = 1,\ \chi_d(p) = -1,\ L_{E_{11},d}(1) = 0\}}$$

Then
$$\lim_{D \to \infty} R_p(D) = \sqrt{\frac{p+1-a_p}{p+1+a_p}}$$

Extension to other compact Lie groups?

eg exceptional Lie groups

eg G_2

14-dimensional group of rank 2 (automorphism group of the octonions) – embedding into $SO(7)$

7-dimensional representation

$$Z(U,\theta) = \det(I - Ue^{-i\theta}) = (1-e^{-i\theta})\tilde{Z}(U,\theta)$$

$$\int_{G_2} |\tilde{Z}(U,0)|^{s}\,dU \underset{?}{=} \frac{\Gamma(3s+7)\Gamma(2s+3)}{\Gamma(2s+6)\Gamma(s+4)\Gamma(s+3)\Gamma(s+2)}$$

Macdonald const. term identity

N. Katz – one-parameter family of L-functions over a finite field whose value distribution in the limit as the size of the field grows is related to G_2.

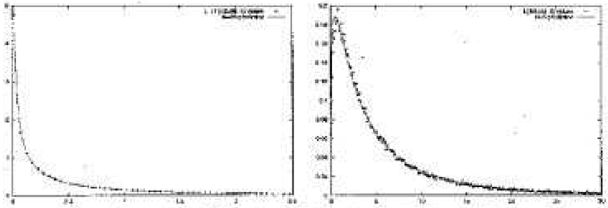
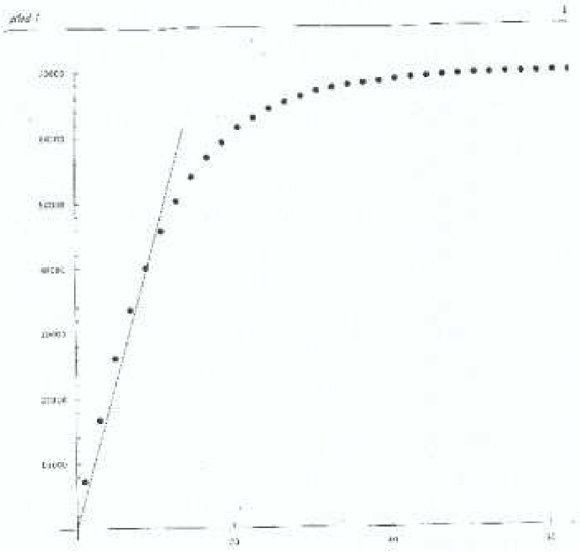


FIGURE 3. The first picture depicts the value distribution of $L_{E_{11}}(1/2, \chi_d)$, for prime $|d|$, $-788299808 < d < -630639845$, even functional equation, compared to $P_O(N, x)$, with $N = 20$. For contrast, we depict, in the second picture, the value distribution of $L(1/2, \chi_d)$ (Dirichlet L-functions) for all fundamental $800000 < |d| < 1000000$. Here, the Katz-Sarnak philosophy predicts a Unitary Symplectic family, and so we compare the data against $P_{USp}(N, x)$, $N = 5$.



Cumulative distribution of $|C_{140}|$

for $|d| < 500{,}000$, (69664 values of d).

plotted with $y = (0.064)\cdot(69864)\cdot X$

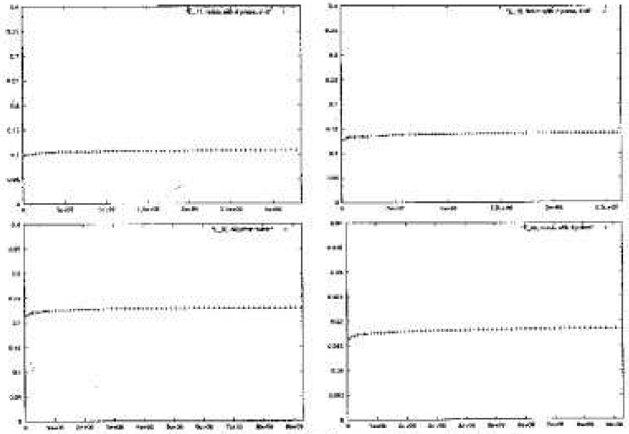


FIGURE 2. Figures in support of Conjecture 1. These depict the l.h.s. of (25) divided by $T^{3/4}(\log T)^{-3/8}$. For the level 11 and 19 curves, we only looked at twists with $d < 0$, d prime, even functional equation. We also depict the l.h.s of (23) divided by $T^{3/4}(\log T)^{11/8}$ for the level 32 curve and odd d. While the pictures are reasonably flat, $\log(T)$ is almost constant for most of the interval in question. The flatness we are observing reflects the main dependance on $T^{3/4}$.

p	conjectured R_p for E_{11}	data for E_{11}	conjectured R_p for E_{19}	data for E_{19}	conjectured R_p for E_{32}	data for E_{32}
3	1.2909944	1.2774873	1.7320508	1.7018241	1	0.99925886
5	0.84515425	0.84938811	0.57735027	0.57825622	1.4142136	1.4113424
7	1.2909944	1.288618	1.1338934	1.134852	1	1.0003445
11		0	0.77459667	0.76491219	1	1.0001457
13	0.74535599	0.73266305	1.3416408	1.3632977	0.63245553	0.61626177
17	1.118034	1.1282072	1.183216	1.196637	0.89442719	0.88962298
19	1	1.000864		0	1	1.0006726
23	1.0425721	1.0470095	1	0.99857962	1	1.0000812
29	1	0.99769402	0.81649658	0.80174375	1.4142136	1.4615854
31	0.80064077	0.78332934	1.1338934	1.143379	1	1.0008405
37	0.92393644	0.91867671	0.9486833	0.94311279	1.0540926	1.0603105
41	1.2126781	1.2400086	1.1547005	1.1683113	0.78446454	0.76494748
43	1.1470787	1.1642671	1.0229915	1.0229106	1	1.0006774
47	0.84515425	0.82819492	1.0645813	1.0708874	1	0.99961502
53	1.118034	1.1332312	0.79772404	0.77715638	0.76696499	0.74137107
59	0.91986621	0.91329134	1.1055416	1.1196252	1	0.99969828
61	0.82199494	0.79865031	1.0162612	1.0199932	1.1766968	1.1996892
67	1.1088319	1.1216776	1.0606602	1.0705574	1	1.0002831
71	1.0425721	1.0497774	0.91986621	0.90939741	1	0.99992715
73	0.94733093	0.94345043	1.099525	1.1110782	1.0846523	1.0950853
79	1.1338934	1.1562237	0.90453403	0.8922209	1	0.99882039
83	1.0741723	1.0854551	0.8660254	0.84732408	1	0.99979996
89	0.84515425	0.82410673	0.87447463	0.85750248	0.89442719	0.88154899
97	1.0741723	1.0877289	0.92144268	0.90867892	0.8304548	0.80611684
101	0.98058068	0.97846254	0.94280904	0.93032086	1.0198039	1.0229168
103	1.1677484	1.1976448	0.87333376	0.855721	1	1.0004009
107	0.84515425	0.82186438	1.183216	1.2153554	1	1.0009252
109	0.91287093	0.89933354	1.1577675	1.1844329	0.94686415	0.94015124
113	0.92393644	0.9146531	0.9486833	0.93966595	1.1313708	1.1534106
127	0.93933644	0.93052596	0.98449518	0.98005032	1	0.99994006
131	1.1470787	1.171545	1.1208971	1.1413931	1	0.99916309
137	1.052079	1.0603352	1.0219806	1.0285831	1.1744404	1.2066518
139	0.93094934	0.91532106	1.0975994	1.1176423	1	1.0006469
149	1.069045	1.0833831	0.86855395	0.84844439	0.91064169	0.89706709

TABLE 1. A table in support of Conjecture 2, comparing R_p v.s. $R_p(T)$ for the three elliptic curves E_{11}, E_{19}, E_{32} (T equal to 333605031, 263273979, 930584451 respectively). More of this data, for $p < 2000$, is depicted in the figures below. The 0 entries for $p = 11$ and $p = 19$ are explained by the fact that we are restricting ourselves to twists with even functional equation, $w_{E\chi_d}(-N) = 1$. Hence for E_{11} and E_{19}, we are only looking at twists with $\chi_d(11) = \chi_d(19) = -1$.

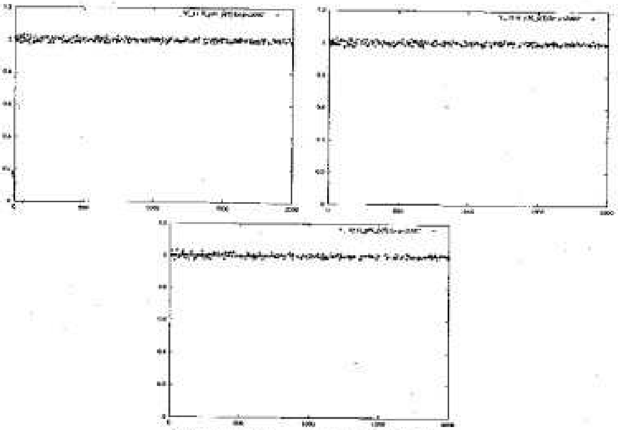


FIGURE 1. Pictures depicting $R_p/R_p(T)$, for $p < 2000$, T as in Table 1.