BROWNIAN MOTION

IN A WEYL CHAMBER

Philippe Biane

Philippe Bougerol

Neil O'Connell

Warwick, May 18, 2004

math.RT/0403171

PITMAN'S THEOREM (1975)

Let $B_t; t \geq 0$ be a real Brownian motion and let

$$I_t = \inf_{0 \leq s \leq t} B_s$$

then $B_t - 2I_t$ three dimensional Bessel process

(i.e. distributed as the norm of a three dimensional Brownien motion).

$M_t = \begin{pmatrix} X_t & U_t - iV_t \\ U_t + iV_t & -X_t \end{pmatrix}$= hermitian matrix valued Brownian motion. The largest eigenvalue of M_t is $\sqrt{X_t^2 + U_t^2 + V_t^2}$= Bessel-3 process.

GUE

Theorem. Barychnikov (PTRF 2001), Gravner, Tracy, Widom (J. Stat. Phys. 2001).

Largest eigenvalue of a GUE matrix is distributed as

$$\max_{1 \geq t_1 \geq \ldots \geq t_{d-1} \geq 0} [W_1(1) - W_1(t_1) + W_2(t_1) - W_2(t_2) + \ldots + W_d(t_{d-1})]$$

where $(W_1, \ldots, W_d)$= d-dimensional Brownian motion.

Consequence of Greene's formula for longest row in a Young diagram obtained by RSK correspondence.

Generalizations

O'Connell and Yor (Elec. Comm. Prob. (2002)) and Bougerol and Jeulin (PTRF (2002)) have given generalizations of this construction: they give max-plus formulas which transform a d dimensional Brownian motion into the eigenvalue process of an hermitian Brownian motion.

O'Connell and Yor: use queuing theory.

Bougerol and Jeulin: use brownian motion on symmetric spaces (construction works for classical series of Lie algebras).

The two formulas look quite different.

Brownian motion in a cone

Let C be a convex cone in R^n.

Let $p_t^0(x,y)$ be the heat kernel on C with Dirichlet boundary conditions = transition probabilities for Brownian motion killed at the boundary of the cone.

Fact: there exists a unique (up to a multiplicative constant) positive p^0-harmonic function on C.

Define the h-process by the transition densities

$$q_t(x,y) = \frac{1}{h(x)} p_t^0(x,y) h(y)$$

we call the corresponding stochastic process "Brownian motion in the cone C".

Examples:

1) $n = 1$ and $C = R_+$, then

$$p_t^0(x,y) = p_t(x,y) - p_t(x,-y) \qquad h(x) = x$$

Brownian motion in the cone C is the three dimension Bessel process.

2) Let W be a Coxeter group and C = fundamental chamber.

$$p_t^0(x,y) = \sum_{w \in W} \varepsilon(w) p_t(x, w(y)) \qquad h(x) = \prod_{\alpha \in R} \alpha(x)$$

If $W = S_{n+1}$ (A_n-type group) then Brownian motion in C is distributed as the eigenvalue process of a Brownian motion on $su_{n+1}(C)$.

We seek a formula for transforming Brownian motion on R^n into Brownian motion in a fundamental chamber.

Generalized Pitman transforms

Let V= euclidian space, $a \in V$, $\langle a, a\rangle = 1$.
Pitman's transform in the direction a is

$$P_a\pi(t) = \pi(t) - 2 \inf_{0\leq s\leq t}\langle \pi(s), a\rangle a$$

this is an idempotent

$$P_a^2 = P_a$$

Fundamental formula.

Let $a, b \in V$, with $\langle a, b\rangle = -\cos\theta$.
Let $n\theta \leq \pi$ then

$$(n \text{ terms}) \qquad P_aP_bP_a\dots\pi(t) = \pi(t)$$
$$- 2 \inf_{t\geq s_1\geq\dots\geq s_n\geq 0}[\frac{\sin\theta}{\sin\theta}\langle\pi(s_1), a\rangle + \frac{\sin 2\theta}{\sin\theta}\langle\pi(s_2), b\rangle + \frac{\sin 3\theta}{\sin\theta}\langle\pi(s_3), a\rangle + \dots]\, a$$
$$- 2 \inf_{t\geq s_1\geq\dots\geq s_{n-1}\geq 0}[\frac{\sin\theta}{\sin\theta}\langle\pi(s_1), b\rangle + \frac{\sin 2\theta}{\sin\theta}\langle\pi(s_2), a\rangle + \frac{\sin 3\theta}{\sin\theta}\langle\pi(s_3), b\rangle + \dots]\, b$$

Braid relations.

If $\theta = \pi/n$ then

$$P_aP_bP_a\dots = P_bP_aP_b\dots \qquad (n \text{ terms})$$

Coxeter groups

Let (W, S) be a finite Coxeter group.

W consists of isometries in some finite dimensional euclidean plane, S is a generating set of hyperplane reflections.

Denote $\frac{\pi}{n_{ij}}$ the angle between hyperplanes of s_i and s_j is then the relations

$$s_i^2 = 1 \qquad (s_i s_j)^{n_{ij}} = 1$$

are defining relations for W.

Choose a fundamental chamber C.

This is a fundamental domain for the action of W, a convex cone bounded by the hyperplanes H_s associated with the reflections $s \in S$.

Let a_s be the unit vector orthogonal to H_s pointing inside C and $P_s \equiv P_{a_s}$.

Pitman operators associated with W

Let $w \in W$ and

$$w = s_1 \dots s_k \qquad s_1, \dots, s_k \in S$$

a decomposition of minimal length.

By the braid relations

$$P_w = P_{s_{i_1}} P_{s_{i_2}} \dots P_{s_{i_k}}$$

depends only on w and not on the reduced decomposition of w.

Let w_0=longest element in W, then P_{w_0} is an idempotent ($P_{w_0}^2 = P_{w_0}$), furthermore for any path π the path $P_{w_0}\pi$ takes values in C.

The main result

Theorem: (B., Bougerol, O'Connell 2004).

1. If W is a Weyl group of type A_n, then P_{w_0} is the same tranform as the one in O'Connell-Yor.

2. If W is a Weyl group of type A_n, B_n, C_n or D_n, then P_{w_0} is the same tranform as the one in Bougerol-Jeulin.

3. If W is a finite Coxeter group, then P_{w_0} transforms Brownian motion in V into a Brownian motion inside the cone C.

A formula for P_w

If W is a Weyl group and $\alpha_i; i \in I$ the simple roots, and $\alpha_i^\vee$ the coroots then one has

$$P_w\pi(t) = \pi(t) - \sum_i \inf_{\substack{i_1\dots,i_r \in S(i,w)\\ t\geq t_1\geq t_2\geq\dots\geq t_r\geq 0}} (\langle\alpha_{i_1}, \pi(t_1)\rangle + \dots \langle\alpha_{i_r}, \pi(t_r)\rangle)\alpha_i^\vee$$

Proof uses

Proof of the braid relations

The proof of the fundamental formula is by induction on n.

The case $n = 3$ is the first nontrivial step. Then the braid relations for $n = 3$ are used in the induction proof.

An "exponential argument" for the $n = 3$ braid relation.

Consider the transformation

$$T_a\pi(t) = \pi(t) + \log\left(\int_0^t e^{-2\langle a,\pi(s)\rangle} ds\right)$$

then by Laplace method

$$P_a\pi = \lim_{\varepsilon\to 0} \varepsilon\, T_a(\frac{1}{\varepsilon}\pi)$$

The braid relation for $\langle a, b\rangle = -\frac{1}{2}$

$$P_aP_bP_a = P_bP_aP_b$$

comes from

$$T_aT_bT_a = T_bT_aT_b$$

which is equivalent to the identity

$$\int_0^t ds \int_0^s dr\ F(r)\frac{G(s)}{G(r)}\frac{H(t)}{H(s)} = \int_0^t ds \int_0^s dr\ F(r)\frac{\tilde{G}(s)}{\tilde{G}(r)}\frac{\tilde{H}(t)}{\tilde{H}(s)}$$

for some positive continuous functions F, G, H, where

$$\tilde{G}(s) = \left(\int_0^s G(r)H(r)^{-1}dr\right)^{-1} G(s)$$

and

$$\tilde{H}(s) = \left(\int_0^s G(r)H(r)^{-1}dr\right) H(s).$$

Proofs of the Brownian property

First proof: uses the decomposition

$$P_{w_0} = P_{a_1} \dots P_{a_q}$$

and iteration of a version of Pitman's theorem (or the fundamental output theorem in queiung theory).

Second proof: uses representation theory and Littelmann path theory (valid only for Weyl groups)