Asymptotic Spectral Approximations

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Plan of the Lecture

Spectral Distribution of Random Matrices

- GUE- or LUE-ensembles:
  Kolmogorov distance between *expected* distribution functions and their limits
- Wigner and sample covariance matrices:
  Kolmogorov distance between *random* distribution functions and their limits

Limit Theorems for Free Convolutions

- Stochastic/analytic definition of free convolutions
- Arithmetic of additive/multiplicative free convolution semigroups
- CLT for additive free convolution

GUE Ensembles

$$
\begin{aligned}
W_N &= \Big(\frac{X_{ij}}{\sqrt{N}}\Big)_{i,j=1,\ldots,N} \quad \text{Hermitean} \\
X_{ij} & \quad \text{i.i.d. },\ i \le j \\
\mathbb{E}X_{ij} &= 0, \quad \mathbb{E}(|X_{ij}|^2) = 1, \quad X_{ij} \ \textbf{Gaussian}
\end{aligned}
$$

$$
\lambda_1 \le \lambda_2 \le \ldots \le \lambda_N \quad \text{eigenvalues of } W_N
$$

density of $\lambda_1, \ldots, \lambda_N$:

$$
\begin{aligned}
\phi_{GUE} &= \frac{1}{Z_N} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^2 \prod_{1 \le j \le N} \exp[-\frac{N}{2}\lambda_j^2] \\
&= c_N \det\Big(\phi_i(\sqrt{N}\lambda_j), i, j = 1, \ldots, n\Big),
\end{aligned}
$$

$\phi_i(x)$, i-th Hermite orthogonal function.

Spectral Distribution Function :

$$F_N(x) := \sharp\{j \le N \mid \lambda_j \le x\}/N$$

$$\lim_N \mathbb{E} F_N(x) = F(x), \quad \text{Wigner's law}$$

$$F(x) := \frac{1}{2\pi}\int_{-2}^{x} \sqrt{4-x^2}\,dx$$

$$\Delta_N(x) := |\mathbb{E}F_N(x) - F(x)|$$

Density of eigenvalues of Hermitean matrices $W_N := \big((X_{lj} + i\,Y_{lj})/\sqrt{N}\big)_{l,j=1,\ldots N}$:

$$\sigma_N(x) := \frac{d}{dx}\mathbb{E}F_N(x) = \frac{1}{\sqrt{N/2}}\sum_{j=0}^{N-1}\varphi_j^2(x\sqrt{N/2}),$$

$$= \sqrt{N}\phi_{N-1}(x\sqrt{N})^2 - \sqrt{N+1}\phi_{N-1}(x\sqrt{N})\phi_{N+1}(x\sqrt{N})$$

where $\varphi_k(x) = k-$th Hermite orthogonal function.

Asymptotic expansions in the bulk and on the boundary for σ_N:

Deift et al. '99, Haagerup and Thorbjornsen '03, Ercolani,McLaughlin '03, Gustavsson '04.

Th. (G.- Tikhomirov '04)

$$\Delta'_N(x) := |\sigma_N(x) - \frac{1}{2\pi}\sqrt{4-x^2}| \leq \frac{c}{(4-x^2)} N^{-1}, \quad c > 0, a > 0$$

provided $x \in [-2 + a\,N^{-2/3}, 2 - a\,N^{-2/3}]$

There exists some constant C

$$\sup_x \Delta_N(x) \leq C\,N^{-1}.$$

optimal bound, conjectured by Silverstein and Bai (1993).

Differential equation for the spectral density

$$\mathcal{A}_N^* \sigma_N(x) := \left((4-x^2)\, D_x + x + \frac{1}{N^2} D_x^3\right)\sigma_N(x) = 0$$

consequences:

Integral representation for the spectral density

$$\sigma_N(x) = \frac{\sqrt{4-x^2}}{2}\sigma_N(0) + \frac{\sqrt{4-x^2}}{N^2}\int_0^x \frac{\sigma_N'''(u)du}{(4-u^2)^{\frac{3}{2}}}$$

and

Integral representation for the spectral distribution

$$\begin{aligned}\Delta_N(x) := \mathbb{E}F_N(x) - F(x) \quad &= \quad \frac{1}{2}\left(\sigma_N(0) - \frac{1}{\pi}\right)\int_0^x \sqrt{4-u^2}\, du \\ &\quad + \frac{1}{N^2}\int_0^x \sqrt{4-u^2}\left(\int_0^u \frac{\sigma_N'''(s)ds}{(4-s^2)^{\frac{3}{2}}}\right) du.\end{aligned}$$

Rates of Convergence for Wigner 'Ensembles'

of Random Matrices

$$W_N = \left(\frac{X_{ij}}{\sqrt{N}}\right)_{i,j=1,\ldots,N} \quad \text{hermitean}, \quad X_{ij} \text{ independent }, \ i \le j$$

$$\mathbb{E}X_{ij} = 0, \quad \mathbb{E}(|X_{ij}|^2) = 1, \ \mathbb{E}|X_{ij}|^4 < \infty$$

$$\sup_x |\mathbb{E}\, F_N(x) - F(x)| = \mathcal{O}(N^{-\alpha}), \ \alpha \le 1$$

Wigner,'55, Bai, '93, '99, '02 $\alpha = 1/4,\ 1/3, 1/2$, 8th moment, Girko, '98, '02 $\alpha = 1/2$, 4th moment

Th. (G.-Tikhomirov, '00)

$$\Delta_N := \sup_x |\mathbb{E}\, F_N(x) - F(x)| \le c \max_{i,j} \Big(\mathbb{E}|X_{ij}|^4\Big)^{1/2} N^{-1/2}$$

Th. (G.-Tikhomirov, '02) (rate of convergence in probability)

$$\Delta_N^* := \mathbb{E} \sup_x |F_N(x) - F(x)| \le c \max_{i,j} \Big(\mathbb{E}|X_{ij}|^{12}\Big)^{1/6} N^{-1/2}$$

Bound for Δ_N is sharp for sparse matrices

$$W_N = N^{-1/2}(X_{ij}\varepsilon_{ij}p_N^{-1/2}), \quad p_N = \mathbb{P}(\varepsilon_{ij}^{(N)} = 1), \quad Np_N \to 0.$$

Rates of Convergence a.s for Wigner 'Ensembles'

$$W_N = \left(\frac{X_{ij}}{\sqrt{N}}\right)_{i,j=1,\dots,N} \text{ hermitean}, \quad X_{ij} \text{ i.i.d. }, \ i \le j$$

$$\mathbb{E}X_{ij} = 0, \quad \mathbb{E}(|X_{ij}|^2) = 1, \ \mathbb{E}|X_{ij}|^q < \infty, \quad q \ge 16$$

Th. (G., Kushmanova, Tikhomirov, '04)

$$\Delta_N^* := \sup_x |F_N(x) - F(x)| = \mathcal{O}(N^{-1/2}(\log N)^{\alpha}), \quad \text{a. s.}, \quad \varepsilon > 0$$

Bai, '02: $\Delta_N^* = \mathcal{O}(N^{-2/5+\varepsilon})$.

Sample Covariance Matrices

Sample of N vectors with $p \sim N$ parameters

$N \times p$ matrix $\mathbf{X}$

Hypothesis:

i.i.d. vectors with independent **Gaussian** components

$$X^T X = \sum \lambda_j e_j e_j^T, \qquad p \times p \text{ sample covariance}$$

Joint distribution of eigenvalues $0 \leq \lambda_1, \ldots, \leq \lambda_p$?

$X^T X$: has Wishart-distribution with joint density

$$\frac{1}{Z_p} \prod_{1 \leq i < j \leq p} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{p} \lambda_j^{N-p} \exp\{-\lambda_j\}$$

'Laguerre-Ensemble'

Approximation of the spectral distribution function

$$F_N(x) := \#\{1 \le j \le N : \lambda_j \le x\}/N,$$

$$\mathbb{E} F_N(x) \to G(x) \text{ (Marchenko-Pastur)}$$

$$G'(x) = \frac{y}{2\pi x}\sqrt{(b-x)(x-a)}\, I(a \le x \le b) + I(1 < y < \infty)(1 - y^{-1})$$

$$a := (1 - y^{-1/2})^2, \quad b := (1 + y^{-1/2})^2$$

$$y := p/N, \ |1 - y| > 0$$

Thm. (G.- Tikhomirov 01, 02, 04) Assume $X_{ij}, i \le j$ independent

$$\gamma_q := \max_{i,j} (\mathbb{E}|X_{ij}|^q)^{2/q} < \infty$$

$$\sup_x |\mathbb{E} F_N(x) - G(x)| \le c\, \gamma_8\, N^{-1/2}$$

$$\mathbb{E} \sup_x |F_N(x) - G(x)| \le c\, \gamma_{12}\, N^{-1/2}$$

Asymptotics for LUE-Ensembles

$$X_{ij}, i \leq j, i = 1, \ldots, p, j = 1, \ldots, N$$

independent standard complex *Gaussian*

XX^T Laguerre unitary ensemble (LUE)
with density

$$\phi(x) = \frac{N}{p} \sum_{j=0}^{p-1} \left(\mathcal{L}_j^{(N-p)}(Nx) \right)^2,$$

$\mathcal{L}_j^{(N-p)}$ orthogonal Laguerre functions. Differential equation for ϕ:

$$(x-a)(b-x)\,\phi'(x) + \left(x - \frac{a+b}{2} \right) \phi(x) + \frac{x\phi''(x)}{N^2} + \frac{x^2\phi'''(x)}{N^2} = 0,$$

Th (G.-Tikhomirov 2003)

$0 < a_1 \leq y = p/N \leq a_2 < 1$

$$\begin{aligned} \Delta_N^{(p)} = \sup_x |\mathbb{E}F_N^{(p)}(x) - G(x)| &\leq C(a_1, a_2) N^{-1} \\ &\leq CN^{-1} \quad \text{for} \quad p = N \end{aligned}$$

Free Convolution: Stochastic Approach

Spectral measures ν_1, ν_2 on $\mathbb{R}$ of compact support.

Discrete approximations $\nu_{A_N} \to \nu_1, \quad \nu_{B_N} \to \nu_2, N \to \infty$, e.g.

$$\mu_{A_N} := \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_{j,N}},$$

$(\lambda_{1,N}, \ldots, \lambda_{N,N})$ eigenvalues of hermitean $N \times N$ matrix A_N.

Similarly, ν_{B_N}: spectral measure of hermitean $N \times N$ matrix B_N.

$U(N)$: $N \times N$ unitary matrices with Haar measure γ_N.

There exists a measure $\nu_1 \boxplus \nu_2$, all $\epsilon > 0$,

$$\lim_N (\gamma_N \times \gamma_N)\{(U_1, U_2) \in U(N)^2 \,:\, d(\nu_{U_1 A_N U_1^* + U_2 B_N U_2^*}, \nu_1 \boxplus \nu_2) > \epsilon\} = 0,$$

d metrizes weak convergence. (Voiculescu 1991, Pastur-Vasilchuk, 2000).

Definition by Voculescu (1986), extension by Maassen (1992), Bercovici,Voiculescu (1993)

Free convolution: Analytic Approach

$\mu \in \mathcal{M}$: distribution on $\mathbb{R}$, on $\mathbb{C}_+$: (upper halfplane)

$$F_\mu(z) := \Big(\int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}\Big)^{-1},$$

is a Nevanlinna function ($\mathbb{C}_+ \to \mathbb{C}_+$), such that $\lim_z F_\mu(z)/z = 1$, $\Re(z)/\Im(z)$ bounded, of class: $\mathcal{F}$.

Key result: Given ν_1, ν_2: exist unique functions $\kappa_j(z) \in \mathcal{F}$, s. th.

$$F_{\nu_1} \circ \kappa_1 = F_{\nu_2} \circ \kappa_2 \in \mathcal{F}, \qquad \kappa_1(z) + \kappa_2(z) \;=\; z + F_{\mu_1}(\kappa_1(z))$$

Consequences:

(1) unique $\nu_1 \boxplus \nu_2$ with $F_{\nu_1 \boxplus \nu_2} = F_{\nu_j} \circ \kappa_j$, $j = 1, 2$,

(Biane 1998),

On $\mathcal{F}$ define: $\phi_\mu(z) := F_\mu^{-1}(z) - z$, (Bercovici, Pata, Biane 1999), Bercovici, Voiculescu 1996

$F_\mu^{-1}(z)$ left-inverse on $\Gamma_{\alpha,\beta} \subset \mathbb{C}_+$: $|x| < \alpha y$, $y > \beta$, with $\alpha, \beta > 0$.

Using $\kappa_j = F_{\nu_j}^{-1} \circ F_{\nu_1 \boxplus \nu_2}$:

(2) $$\phi_{\nu_1 \boxplus \nu_2} = \phi_{\nu_1} + \phi_{\nu_2}.$$

N-fold (identical) convolutions:

given measure μ and N: exist a unique $\kappa \in \mathcal{F}$ with

$$\begin{aligned} N\kappa(z) &= z + (N-1)F_\mu(\kappa(z)) \\ F_{\mu\boxplus\ldots\boxplus\mu}(z) &:= F_\mu(\kappa(z)) \\ \phi_{\mu\boxplus\ldots\boxplus\mu} &= N\,\phi_\mu \end{aligned}$$

Examples:

$$\begin{aligned} F_{\delta_a}(z) &= z - a, \quad \phi_{\delta_a}(z) = a \\ F_{Cauchy(b)}(z) &= z + ib, \quad \phi_{Cauchy(b)}(z) = -i\,b \\ F_{Wigner}(z) &= 2\sigma^2/(z - \sqrt{4\sigma^2 - z^2}), \quad \phi_{Wigner}(z) = \sigma^2/z \\ F_{Poisson(\lambda)}(z) &= \frac{1}{2}(z + 1 - \lambda + \sqrt{(z+1-\lambda)^2 - 4(z+\lambda)}, \quad \phi_{Poisson(\lambda)}(z) = \lambda\frac{z+1}{z-1} \end{aligned}$$

Multiplicative Free Convolution on $\mathbb{R}_+$

Krein class $\mathcal{K}$: $f, f/z : \mathbb{C}_+ \to \mathbb{C}_+$

$\mu \in \mathcal{M}_+$: distribution on $\mathbb{R}_+$, for $z \in \mathbb{C} \setminus \mathbb{R}_+$

$$\begin{aligned} \psi_\mu(z) &:= \int_{\mathbb{R}_+} \frac{zt}{1-zt}\mu(dt), \quad R_\mu := \frac{\psi_\mu}{1+\psi_\mu}, \\ \lim_{x\to 0} R_\mu(x) &= 0, \quad \text{class: } \mathcal{K}_* \subset \mathcal{K} \\ \Sigma(z) &:= R_\mu^{-1}(z)/z \end{aligned}$$

μ_1, μ_2: distribution on $\mathbb{R}_+$: exist unique $\xi_1, \xi_2 \in \mathcal{K}_*$:

$$\begin{aligned} R_{\mu_1} \circ \xi_1 = R_{\mu_2} \circ \xi_2 &=: R_{\mu_1 \boxtimes \mu_2}, \\ \xi_1(z)\xi_2(z) &= z\, R_{\mu_j}(\xi_j(z)), \quad \text{hence,} \end{aligned}$$

$$\Sigma_{\mu_1 \boxtimes \mu_2} = \Sigma_{\mu_1}\Sigma_{\mu_2}, \quad \text{and for some } \xi$$

$$\begin{aligned} \xi(z)^N &= z\, R_\mu(\xi(z))^{N-1}, \quad R_{\mu\boxtimes\ldots\boxtimes\mu} = R_\mu \circ \xi,\ z \in \mathbb{C}_+ \\ \Sigma_{\mu\boxtimes\ldots\boxtimes\mu} &= \Sigma_\mu^N \end{aligned}$$

Multiplicative Free Convolution on S^1

$D := \{|z| < 1\}$, $\mathcal{S}$, Schur: $f : D \to \bar{D}$,
$\mu \in \mathcal{M}_*$: distribution on S^1, $\int x d\mu(x) \neq 0$

$$\begin{aligned} \psi_\mu(z) &:= \int_{S^1} \frac{zt}{1-zt}\mu(dt), \quad Q_\mu := \frac{\psi_\mu}{1+\psi_\mu}, \\ Q_\mu(0) &= 0,\ Q'_\mu(0) \neq 0 \quad \text{class: } \mathcal{S}_* \subset \mathcal{S} \\ \Sigma(z) &:= Q_\mu^{-1}(z)/z \end{aligned}$$

μ_1, μ_2 on S^1, $\int x d\mu(x) \neq 0$, exist unique $\xi_1, \xi_2 \in \mathcal{S}_*$:

$$\begin{aligned} Q_{\mu_1} \circ \xi_1 = Q_{\mu_2} \circ \xi_2 &=: Q_{\mu_1 \boxtimes \mu_2}, \\ \xi_1(z)\xi_2(z) &= z Q_{\mu_j}(\xi_j(z)), \quad \text{hence,} \\ \Sigma_{\mu_1 \boxtimes \mu_2} &= \Sigma_{\mu_1}\Sigma_{\mu_2}, \quad \text{and for some } \xi \end{aligned}$$

$$\begin{aligned} \xi(z)^N &= z Q_\mu(\xi(z))^{N-1}, \quad Q_{\mu\boxtimes\ldots\boxtimes\mu} = Q_\mu \circ \xi,\ z \in \mathbb{C}_+ \\ \Sigma_{\mu\boxtimes\ldots\boxtimes\mu} &= \Sigma_\mu^N \end{aligned}$$

Arithmetic of Convolutions

Khintchine theorems for classical convolution $(\mathcal{M}, *)$,

- **Khintchine-CLT**: A distribution is infinitely divisible *if and only if* it is the weak convolution limit of a triangular array of asymptotically infinitesimal measures

- **Arithmetic**: (improper: δ_a). Let $\mu \in \mathcal{M}$:
  1. *class ID:* μ is indecomposable (improper factors only)
  2. *class I_1:* μ is decomposable and has **an** indecomposable factor
  3. *class I_0:* μ is decomposable and has **no** indecomposable factor

- **Decomposition:** $\mu = \nu * \nu_1 * \nu_2 * \cdots, \ \nu \in I_0, \nu_j \in ID$, countable convolution

Gaussian, Poisson in I_0: I_0 nontrivial
$I_0 \subset$ infinitely divisible distributions.

Free Convolution Arithmetic

Levy-Khintchine representation analogues

- $(\mathcal{M}, \boxplus)$, $(\mathcal{M}_+, \boxtimes)$, $(\mathcal{M}_*, \boxtimes)$: (Bercovici, Voiculescu '92,'93, Voiculescu '87)

Khintchine-CLT

- $(\mathcal{M}, \boxplus)$, $(\mathcal{M}_+, \boxtimes)$ (i.i.d.), (Bercovici, Pata '99, '00),
- $(\mathcal{M}, \boxplus)$, $(\mathcal{M}_+, \boxtimes)$, $(\mathcal{M}_*, \boxtimes)$: (Chistyakov-G. '04).

Arithmetic

- $(\mathcal{M}, \boxplus)$, $(\mathcal{M}_+, \boxtimes)$, $(\mathcal{M}_*, \boxtimes)$:
  1. $I_0 = \{\delta_a\}$, (Bercovici, Voiculescu '95: Wigner $\notin I_0$)
  2. "finite prime" support μ's are indecomposable
  3. ID-class is dense in the weak topology.
- $(\mathcal{M}, \boxplus)$: finite support μ's are indecomposable

CLT for Additive Free Convolution

$\mu_{N1}, \ldots, \mu_{Nk_N}$ triangular array of distributions. Let $\epsilon > 0$

$$\lim_{N} \max_{1 \le k \le k_N} \mu_{Nk}(\{|u| > \epsilon\}) = 0$$

- Weak limits of $\mu^{(N)} := \delta_{a_N} \boxplus \mu_{N1} \boxplus \ldots \boxplus \mu_{Nk_N}$ are $\boxplus$-infinitely divisible distributions.
- Conditions on μ_{Nk}, a_N for convergence of $\mu^{(N)}$ as in classical probability.

Rate of Convergence in the CLT

w: Wigner half-circle distribution: $\frac{1}{2\pi}\sqrt{(4-x^2)_+}$

Δ: Kolmogorov distance

Th. (Chistyakov-G. 2004)

$$m_k(\mu) := \int u^k \mu(du), \quad \mu_N((-\infty,x]) := \mu((-\infty, x\sqrt{N}])$$

$$m_1(\mu) = 0, \quad m_2(\mu) = 1$$

$$\Delta(\mu_N^{N\boxplus}, w) \leq c\,\frac{|m_3(\mu)| + m_4(\mu)N^{-1/4}}{N^{1/2}},$$

provided $m_4(\mu) < \infty$.

Sharp bound: exist two-point distribution μ

$$\Delta(\mu_N^{N\boxplus}, w) \geq cN^{-1/2}$$

Representations of Infinitely Divisible Distributions

$$
\begin{aligned}
\phi_\nu(z) &= \; = \alpha + \int \frac{1+u\,z}{z-u}\nu(du), \;\; z \in \mathbb{C}_+ \\
\Sigma_\nu(z) &= \exp\Big\{a - bz + \int_{\mathbb{R}_+} \frac{1+u\,z}{z-u}\nu(du)\Big\}, \;\; 0 < arg(z) < 2\pi \\
\Sigma_\nu(z) &= \exp\Big\{i\,a - \int_{S^1} \frac{1+u\,z}{1-u\,z}\nu(du)\Big\}, \;\; z \in D.
\end{aligned}
$$

Bounds for Supremum-Distances

Let

$$\mathbf{R}(z) = (\mathbf{W} - z\mathbf{I})^{-1}, \quad m_N(z) = \frac{1}{N}\operatorname{Tr}\mathbf{R}(z), \qquad s_N(z) = \mathbb{E}m_N(z),$$

and $s(z)$ be Stieltjes transform of the semi-circle distribution.

Using

$$\sup_x |F_N(x) - F(x)| \le C_1 \int_{-\infty}^{\infty} |(m_N(u+iV) - s(u+iV)|du + C_2 v + C_3 \varepsilon^{3/2}$$

$$+ C_1 \sup_{x\in I'_\varepsilon} \Big|\Im\Big\{\int_v^V (m_N(x+iu) - s(x+iu))du\Big\}\Big|$$

$$\sup_{x\in I_\varepsilon} |m_N(x+iv) - \mathbb{E}m_N(u+iv)| \le |m_N(x_0+iv) - \mathbb{E}m_N(x_0+iv)|$$

$$+ \int_{x\in I_\varepsilon} |m'_N(u+iv) - \mathbb{E}m'_N(u+iv)|du$$

and

$$m'_N(z) = \frac{1}{N}\operatorname{Tr}\mathbf{R}^2(z).$$

Bounds for Supremum-Distances

$$\mathbb{E}\sup_x |F_N(x) - \mathbb{E}F_N(x)| \leq$$

$$C_1 \int_{-\infty}^{\infty} |(s_N(u+iV) - s(u+iV)|du + C_2\, v + C_3\, \varepsilon^{3/2}$$

$$+C_4 \sup_{x\in I'_\varepsilon} \Big|\Im\Big\{\int_v^V (s_N(x+iu) - s(x+iu)du\Big\}\Big|$$

$$+C_1 \int_{-\infty}^{\infty} \mathbb{E}|\frac{1}{N}(\mathrm{Tr}\,\mathbf{R}(u+iV) - \mathbb{E}\,\mathrm{Tr}\,\mathbf{R}(u+iV))|du$$

$$+C_1 \int_v^V \mathbb{E}\Big|\frac{1}{N}\,\mathrm{Tr}\,\mathbf{R}(x_0+iu) - \frac{1}{N}\mathbb{E}\,\mathrm{Tr}\,\mathbf{R}(x_0+iu))\Big|du$$

$$+C_2 \int_v^V \int_{x\in I_\varepsilon} \mathbb{E}\Big|(\frac{1}{N}\,\mathrm{Tr}\,\mathbf{R}^2(x+iu) - \mathbb{E}\frac{1}{N}\,\mathrm{Tr}\,\mathbf{R}^2(x+iu))\Big|dxdu$$

Stieltjes Transforms and Martingales

$$
\begin{aligned}
\mathbf{W} &= (X_{ij}/\sqrt{N})_{i,j=1,\ldots,N} \\
\mathbf{A} &:= \mathbf{W} - z\mathbf{I} =: (a_{i,j}), \quad z \in \mathbf{C} \\
&= \begin{pmatrix} \mathbf{A}_N & \alpha_N \\ \\ \alpha_N^T & a_{N,N} \end{pmatrix},
\end{aligned}
$$

$$
\alpha_N^T = (a_{N,1}, \ldots, a_{N,N-1}), \quad \mathbf{A}_N = (a_{i,j},\, i,j \le N-1)
$$

$$
\begin{pmatrix} \mathbf{I} & 0 \\ \\ -\alpha_N^T \mathbf{A}_N^{-1} & 1 \end{pmatrix} \mathbf{A} = \begin{pmatrix} \mathbf{A}_N & \alpha_N \\ \\ 0 & a_{N,N} - \alpha_N^T \mathbf{A}_N^{-1} \alpha_N \end{pmatrix}
$$

$$
1 \times \det \mathbf{A} = \det \mathbf{A}_N \times (a_{N,N} - \alpha_N^T \mathbf{A}_N^{-1} \alpha_N)
$$

$$
\begin{aligned}
(A^{-1})_{N,N} &= \frac{\det A_N}{\det A} = \frac{1}{W_{N,N} - z - \alpha_N^T A_N^{-1} \alpha_N} \\
s_N(z) &= \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}(A^{-1})_{k,k}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\,\mathbb{E}_{\alpha_N}\, \alpha_N^T A_N^{-1} \alpha_N &= \sum_{j,k} (\mathbb{E}\, W_{j,N} W_{k,N})\, \mathbb{E}\,(A_N^{-1})_{j,k} \\
&= \frac{1}{N}\, \mathbb{E}\, \mathrm{tr}\, \mathrm{A_N^{-1}} \\
&= s_N(z) + \varepsilon_N
\end{aligned}
$$

$$
s_N(z) = -\frac{1}{z + s_N(z)} + \delta_N(z)
$$

Show

$$
|\delta_N(z)| \le \frac{c}{N\,|\Im(z)|^3} \left| s_N + z \right|^{-2}
$$

Approximate Limit Equations

$$s_N(z) = -\frac{1}{z + s_N(z)} + \delta_N(z)$$

Show

$$|\delta_N(z)| \leq \frac{c}{N\,|\Im(z)|^3}\,\big|s_N + z\big|^{-2}$$

approximate quadratic equation :

$$\Im(z + s_N(z)) \quad = \quad \frac{\Im(z + s_N(z))}{|z + s_N(z)|^2} + \Im(z + \delta_N(z))$$

$$\Im(z + s_N(z))\big(1 - \frac{1}{|z + s_N(z)|^2}\big) \quad = \quad \Im(z + \delta_N(z)) > 0$$

$$\text{implies} \quad |z + s_N(z)| \quad > \quad 1$$

Case

$$\begin{aligned} \Im(z+\delta_N(z)) &= 0 \quad \text{show} \\ |\delta_N(z)| &\le \frac{c}{N\,|\Im(z)|}, \quad \text{for} \\ |\Im z| &> c^{1/2}N^{-1/2} \end{aligned}$$

$$\begin{aligned} \text{Hence} \quad c^{1/2}N^{-1/2} &< |\Im(z)| = |\Im(\delta_N(z))| \le |\delta_N(z)| \\ &\le \frac{c}{N\,|\Im(z)|} \end{aligned}$$

contradiction. Hence

$$\Im(z+\delta_N(z)) \neq 0$$

Marchenko-Pastur

$$W_{i,j} = \frac{1}{N}\sum_{k=1}^{N} X_{i,k}X_{j,k}, \quad i,j = 1,\dots,p$$

$$W_{p,p} \approx 1$$

$$y = p/N$$

$$\mathbb{E}_{\alpha_p}\alpha_p^T A_p^{-1}\alpha_p = \frac{1}{N}\mathrm{tr}\left(\frac{\mathrm{W_p}}{-\mathrm{z\,I}}\right)$$

$$= y\,z\,\mathrm{tr}\left(\frac{1}{\mathrm{W_p}-\mathrm{z\,I}}\right)+\mathrm{y}$$

$$s_p = \frac{1}{y+z-1+y\,z\,s_p(z)}$$

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