Differential Equations for Dyson Processes

Joint work with Craig Tracy

Start with $n \times n$ GUE matrix, let entries independently undergo Ornstein-Uhlenbeck diffusion. (Dyson Brownian motion.) Eigenvalues describe n curves. *Hermite Process*.

Let $n \to \infty$, scale near the top. Infinitely many curves, *Airy process*. Top curve $A(\tau)$. (Prähoffer-Spohn, Johansson.)

Scale in the bulk, *sine process*. Evolution of singular values of complex matrices leads to *Laguerre process*; scaling this at bottom edge gives *Bessel process*.

Differential equations that determine the probabilities

$$\Pr\left(A(\tau_1) \leq \xi_1, \ldots, A(\tau_m) \leq \xi_m\right).$$

Tracy-Widom: $\xi_k = \eta_k + \xi, \quad \eta_k$ fixed. System of ODEs in ξ. (τ_k parameters.)

Adler-van Moerbeke: $m = 2$, $\tau_2 - \tau_1 = t$. PDE in t, ξ_1, ξ_2. (Scaled PDE for Hermite process.)

Present result. X_k finite unions of intervals. Interested in probability that for $k = 1, \ldots, m$ no curve passes through X_k at time τ_k. For Airy process when $X_k = \chi_{(\xi_k, \infty)}$ this is

$$\mathrm{Pr}\,(A(\tau_1) \le \xi_1, \ldots, A(\tau_m) \le \xi_m).$$

Found systems of PDEs with endpoints of the X_k as independent variables whose solutions determine this probability. Did this for all but Laguerre process.

Airy process simplest. Probability given as Fredholm determinant of *extended Airy kernel*, an $m \times m$ matrix kernel. Entries $L_{ij}(x, y)$ given by

$$\int_0^\infty e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz \qquad \text{if } i \ge j,$$

$$-\int_{-\infty}^0 e^{-z\,(\tau_i - \tau_j)}\,\mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,dz \quad \text{if } i < j.$$

$$K_{ij}(x, y) = L_{ij}(x, y)\,\chi_{X_j}(y).$$

Probability equals $\det\,(I - K)$.

Case $X_k = (\xi_k, \infty)$. Set $R = K\,(I - K)^{-1}$.

$$\partial_{\xi_k} \log \det\,(I - K) = R_{kk}(\xi_k, \xi_k).$$

Unknowns will be five matrix functions of the ξ_k. First is

$$r_{ij} = R_{ij}(\xi_i, \xi_j).$$

To define others, let

$$A = \text{diag}\,(\text{Ai}), \quad \chi = \text{diag}\,(\chi_{(\xi_k,\infty)}),$$

$$Q = (I-K)^{-1}A, \quad \tilde{Q} = A\chi(I-K)^{-1}.$$

Other unknowns are

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}'_{ij}(\xi_j).$$

Define r_x and r_y by

$$(r_x)_{ij} = (\partial_x R)_{ij}(\xi_i, \xi_j), \quad (r_y)_{ij} = (\partial_y R)_{ij}(\xi_i, \xi_j).$$

These are not unknowns.

Set $\xi = \text{diag}\,(\xi_k)$. Equations are

$$\begin{aligned}
dr &= -r\,d\xi\,r + d\xi\,r_x + r_y\,d\xi,\\
dq &= d\xi\,q' - r\,d\xi\,q,\\
d\tilde{q} &= \tilde{q}'\,d\xi - \tilde{q}\,d\xi\,r,\\
dq' &= d\xi\,\xi\,q - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q',\\
d\tilde{q}' &= \tilde{q}\,\xi\,d\xi - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi.
\end{aligned}$$

Have to show diagonal entries of $r_x + r_y$ and off-diagonal entries of r_x and r_y are expressible in terms of the unknowns. Here is where the τ_k enter. Let $\tau = \text{diag}\,(\tau)$. Commutators

$$[D,\, L] = -A \otimes A + [\tau, L],$$

$$[D^2 - M,\, L] = 0.$$

¿From these can derive

$$r_x + r_y = -q\,\tilde{q} + r^2 + [\tau,\, r],$$

$$[\tau,\, r_x - r_y] = q'\,\tilde{q} - q\,\tilde{q}' + [r,\, r_x + r_y] + [\xi,\, r].$$

Case $m = 1$. Then $\tilde{q} = q,\ \tilde{q}' = q',\ r = R(\xi,\, \xi)$ and equations give

$$\frac{d^2q}{d\xi^2} = \xi\, q + 2\,(r^2 - r_x - r_y)q.$$

Thus

$$\frac{d^2q}{d\xi^2} = \xi\, q + 2\, q^3.$$

Painlevé II. Also,

$$\frac{d^2}{d\xi^2} \log\, \det\,(I - K) = \frac{dr}{d\xi} = -r^2 + r_x + r_y = -q^2.$$

Hermite process. Extended Hermite kernel (Johansson, Eynard-Mehta) has entries $L_{ij}(x,y)$ given by

$$\sum_{k=0}^{n-1} e^{(k-n)\,(\tau_i-\tau_j)}\,\varphi_k(x)\,\varphi_k(y) \qquad \text{if } i \geq j,$$

$$-\sum_{k=n}^{\infty} e^{(k-n)\,(\tau_i-\tau_j)}\,\varphi_k(x)\,\varphi_k(y) \quad \text{if } i < j.$$

Set

$$\varphi = (2n)^{1/4}\,\varphi_n, \quad \psi = (2n)^{1/4}\,\varphi_{n-1},$$

and define

$$Q = (I-K)^{-1}\,\varphi, \quad P = (I-K)^{-1}\,\psi,$$

$$\tilde{Q} = \varphi\chi\,(I-K)^{-1}, \quad \tilde{P} = \psi\chi\,(I-K)^{-1}.$$

Unknowns $r_{ij} = R_{ij}(\xi_i,\, \xi_j)$ and $q,\ \tilde{q},\ p,\ \tilde{p}$ given by

$$q_{ij} = Q_{ij}(\xi_i), \quad \tilde{q}_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$p_{ij} = P_{ij}(\xi_i), \quad \tilde{p}_{ij} = \tilde{P}_{ij}(\xi_j),$$

$$q'_{ij} = Q'_{ij}(\xi_i), \quad \tilde{q}'_{ij} = \tilde{Q}_{ij}(\xi_j),$$

$$p'_{ij} = P'_{ij}(\xi_i), \quad \tilde{p}'_{ij} = \tilde{P}'_{ij}(\xi_j).$$

Equations

$$
\begin{aligned}
dr &= -r\,d\xi\,r + d\xi\,r_x + r_y\,d\xi, \\
dq &= d\xi\,q' - r\,d\xi\,q, \\
d\tilde{q} &= \tilde{q}'\,d\xi - \tilde{q}\,d\xi\,r, \\
dq' &= d\xi\,(\xi^2 - 2n - 1)\,q - (r_x\,d\xi + d\xi\,r_y)\,q + d\xi\,r\,q', \\
d\tilde{q}' &= \tilde{q}\,(\xi^2 - 2n - 1)\,d\xi - \tilde{q}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{q}'\,r\,d\xi, \\
dp &= d\xi\,p' - r\,d\xi\,p, \\
d\tilde{p} &= \tilde{p}'\,d\xi - \tilde{p}\,d\xi\,r, \\
dp' &= d\xi\,(\xi^2 - 2n + 1)\,p - (r_x\,d\xi + d\xi\,r_y)\,p + d\xi\,r\,p', \\
d\tilde{p}' &= \tilde{p}\,(\xi^2 - 2n + 1)\,d\xi - \tilde{p}\,(d\xi\,r_y + r_x\,d\xi) + \tilde{p}'\,r\,d\xi.
\end{aligned}
$$

Commutators with $D^2 - M^2$ and $e^{\pm\tau}(D \mp M)$.

Case $m = 1$. Can eliminate q and p, arrive at

$$\frac{d^3 r}{d\xi^3} = 4(\xi^2 - 2n)\frac{dr}{d\xi} - 4\xi r - 6\left(\frac{dr}{d\xi}\right)^2 .$$

Integrates to Painlevé IV.