Granular Bosonization (or Fyodorov meets SUSY)

Warwick University (May 18, 2004)

- Bosonization of Dirac fermions
- Fyodorov‘s method for inverse determinants
- Existence of extended states in a 3d granular model
- Generalizations of Fyodorov’s method: fermionic and supersymmetric variants
- Applications

Random Matrix Methods

Methods based on the joint probability density for the eigenvalues of a random matrix:

- Orthogonal polynomials + Riemann-Hilbert techniques
- (Scaling limit) reduction to integrable PDE's (Painleve-type)

In contrast, superanalytic methods apply to band random matrices, granular models, random Schroedinger operators etc.

- Hermitian (or Hamiltonian) disorder:
  Schaefer-Wegner method (1980)
  Fyodorov's method (2001)

- Unitary (scattering, time evolution) disorder:
  color-flavor transformation (1996)
  Howe pairs and duality

Bosonization of Dirac fermions

Functional integral: $Z = \int \exp\left(- \int_{\Sigma} L \, d^2 x \right)$

Lagrangian for Dirac fields:

$$L_{\mathrm{F}} = \bar{y}\, g^{m} (\mathrm{i}\partial_{m} + A_{m}) y \; + m \bar{y} y$$

Lagrangian for boson fields:

$$L_{\mathrm{B}} = \frac{1}{8p} \partial^{m} j \, \partial_{m} j \; + e^{mn} A_{m} \partial_{n} j \; + m \cos j$$

Bosonization is the statement $Z_{\mathrm{F}}[A] = Z_{\mathrm{B}}[A]$.

Fyodorov's Method for Inverse Determinants

Suppose we want to average inverse determinants

w.r.t. GUE_N measure $dm = \mathrm{const}\times \mathrm{e}^{-\mathrm{Tr}H^2/2l^2} dH.$

$$\left\langle \prod_{a=1}^{n} \mathrm{Det}^{-1}(E_a + \mathrm{i}\epsilon - H) \right\rangle_{\mathrm{GUE}_N}$$

$$= \left\langle \prod_{a=1}^{n} \int \mathrm{e}^{\mathrm{i}(\bar{j}_a, (E_a + \mathrm{i}\epsilon - H) j_a)} \right\rangle_{\mathrm{GUE}_N}$$

$$= \int \mathrm{e}^{\mathrm{i}\sum_a (E_a + \mathrm{i}\epsilon)(\bar{j}_a, j_a)} \; \mathrm{e}^{-(l^2/2)\sum_{ab}(\bar{j}_a, j_b)(\bar{j}_b, j_a)}$$

$$= \int \mathrm{e}^{\mathrm{i}\mathrm{Tr}(E + \mathrm{i}\epsilon)M} \; \mathrm{e}^{-(l^2/2)\mathrm{Tr}M^2} \mathrm{Det}^{N-n}(M)\, dM.$$

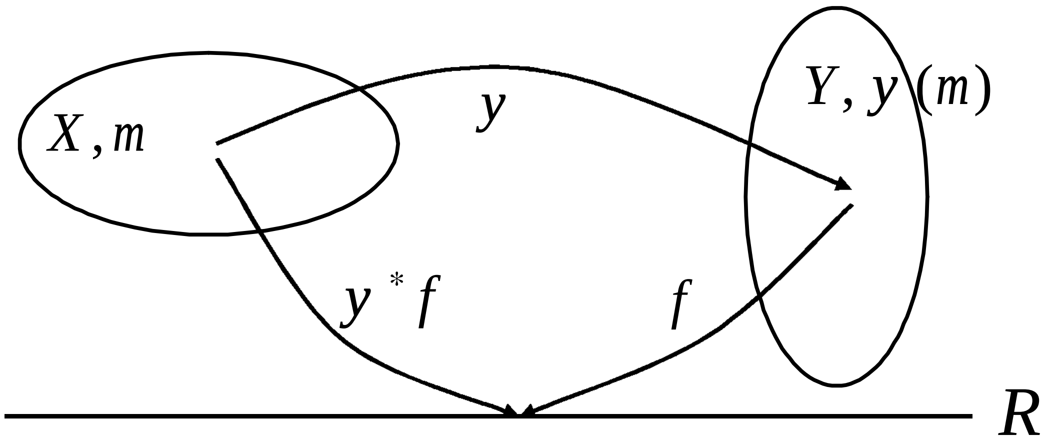
push forward by $M_{ab} = (\bar{j}_a, j_b)$; $E = \mathrm{diag}(E_1, \ldots, E_n)$

Push Forward

Mapping $y : X \rightarrow Y$

Pull back of functions: $y^* f = f \circ y$

Push forward of distributions: $y(m)[f] = m[f^* y]$



Application to Fyodorov's Case

$$X := \mathrm{Hom}(C^n, C^N) \;\to\; \mathrm{Herm}^{\geq 0}(C^n) =: Y$$

$$j \;\mapsto\; j^* j =: M$$

$$\prod_{i=1}^{N} \prod_{a=1}^{n} dj_{i,a}\, d\bar{j}_{i,a} \;\mapsto\; \mathrm{Det}^{N-n}(M)\, dM\,.$$

Normalization:

$$\int_X \mathrm{e}^{-\mathrm{Tr}\, j^* j} \prod dj\; d\bar{j} \;=\; \int_Y \mathrm{e}^{-\mathrm{Tr}\, M}\, \mathrm{Det}^{N-n}(M)\, dM$$

Note the requirement $N \geq n$.

Invariant-theoretic Viewpoint

Identify $\mathrm{GL}(n,C)/\mathrm{U}(n)$ with $\mathrm{Herm}^{>0}(C^n)$ by

Cartan embedding $T \cdot \mathrm{U}(n) \mapsto TT^*$

$\mathrm{GL}(n,C)$ acts by $M \mapsto TMT^*$.

$\mathrm{GL}(n,C)$-invariant measure: $dm(M) = \mathrm{Det}^{-n}(M)\, dM$.

Compare the transformation behavior of $\prod dj\ d\bar{j}$

under $j \mapsto j \circ T$, $j^* \mapsto T^* \circ j^*$

with the transformation behavior of $\mathrm{Det}^N(M)\, dm(M)$

under $T \mapsto T^* M\, T$. They are the same.

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$$\left\langle \prod_{a=1}^{n} \mathrm{Det}^{-1}(E_a + \mathrm{i}e - H) \right\rangle_{\mathrm{GUE}_N}$$

$$= \left\langle \prod_{a=1}^{n} \int \mathrm{e}^{\mathrm{i}(\bar{j}_a, (E_a + \mathrm{i}e - H) j_a)} \right\rangle_{\mathrm{GUE}_N}$$

$$= \int \mathrm{e}^{\mathrm{i} \sum_a (E_a + \mathrm{i}e)(\bar{j}_a, j_a)} \; \mathrm{e}^{-(l^2/2N) \sum_{ab} (\bar{j}_a, j_b)(\bar{j}_b, j_a)}$$

$$= \int \mathrm{e}^{\mathrm{i} N \,\mathrm{Tr}(E + \mathrm{i}e) M} \; \mathrm{e}^{-N(l^2/2) \mathrm{Tr} M^2} \mathrm{Det}^N(M)\, dm(M).$$

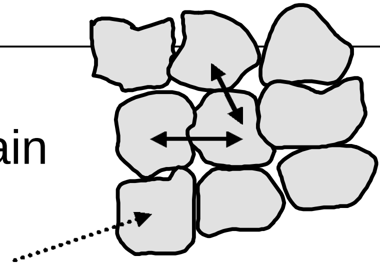
push forward by $M_{ab} = (\bar{j}_a, j_b)/N$; $E = \mathrm{diag}(E_1, ..., E_n)$

Granular Model (deformed)

("Weakly coupled GUEs")

disordered metallic grain

with N electron states



prob. measure $dm(H) = c\,|\,\mathrm{Det}(E + \mathrm{i}e - H)\,|^{-2}\; e^{-\sum_{ij} J^{-1}{}_{ij} \mathrm{Tr}\,\Pi_i H \Pi_j H}\, dH$

$$\left\langle |\,(E + \mathrm{i}e - H)^{-1}(0,0)\,|^2 \right\rangle_m = \int \prod\nolimits_k \mathrm{Det}^N(M_k)\, dn(M_k) \times$$

$$\times M_{0,11} M_{0,22}\; e^{-\sum_{ij} J_{ij} \mathrm{Tr}(sM_i sM_j)/4 - \sum_k \mathrm{Tr}(e - \mathrm{i}sE) M_k}$$

2×2 Hermitian matrices $M_i > 0$; $s = \mathrm{diag}(1,-1)$

Noncompact global symmetry (at $e = 0$):

$$M_i \mapsto T M_i T^* \,, \qquad T^* = s T^{-1} s \in \mathrm{SU}(1,1)$$

Origin of Hyperbolic Symmetry

$$|\,\mathrm{Det}(E+\mathrm{i}e-H)\,|^{-2}=$$

$$\int e^{+\mathrm{i}(\bar{j}_1,(E+\mathrm{i}e-\mathrm{H})j_1)}\,e^{-\mathrm{i}(\bar{j}_2,(E-\mathrm{i}e-\mathrm{H})j_2)}$$

Note the sign change forced by the requirement of convergence of the integral. Hence we are dealing with a quadratic form of indefinite signature.

Sigma Model Approximation

SU(1,1) orbit $M = TT^* \equiv x$ is H^2 (2-hyperboloid)

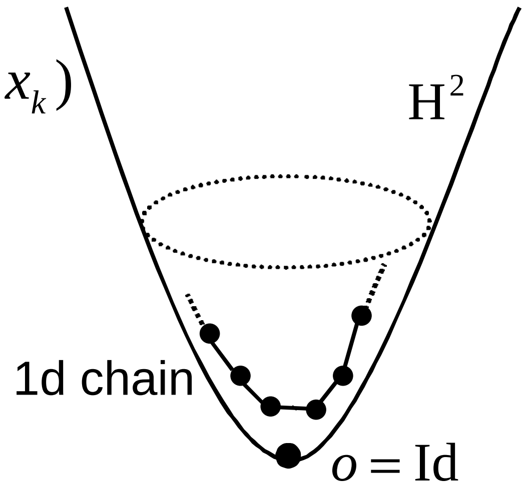
Restrict to critical manifold $\mathrm{H}^2 \times \ldots \times \mathrm{H}^2$

(by eliminating the massive modes).

$$\left\langle |(E + \mathrm{i}\varepsilon - H)^{-1}(0,0)|^2 \right\rangle_m \cong \left\langle \cosh^2 \circ \operatorname{dist}(x_0, o) \right\rangle_S$$

Gibbs measure $e^{-S} \prod_k d\mathrm{vol}(x_k)$

$$S = \beta \sum\nolimits'_{ij} \cosh \circ \operatorname{dist}(x_i, x_j) + \varepsilon \sum\nolimits_k \cosh \circ \operatorname{dist}(x_k, o)$$



Spontaneous Symmetry Breaking

Theorem (Spencer, MRZ):

$$\left\langle \cosh^2 \circ \operatorname{dist}(x_0, o) \right\rangle_S \leq \text{const} \quad \text{if} \quad e \cdot \text{vol} \geq 1$$

and if $d \geq 3$ and b is not too small.

Remark: This bound means that the 'gas' stays near the origin. In the infinite-volume limit it implies regularity of the Green's function in e (in sigma model approximation) and is consistent with the existence of extended states.

Proof: Use the Iwasawa decomposition $G = NAK$ for SU(1,1). Integrate out the nilpotent degrees of freedom, resulting in convex action for the torus variables. Apply the Brascamp-Lieb inequality.

Generalizing Fyodorov's Method

Bosonic variant:

$$M = (M_{ab}) = (\bar{j}_a \cdot j_b)$$ positive Hermitian

$$\int F(\bar{j} \cdot j)\, dj\, d\bar{j} = \int_{M>0} F(M)\, \mathrm{Det}^N(M)\, dm(M)$$

with $dm(M)$ invariant under $M \to TMT^*$

Fermionic variant:

Scalar products of anticommuting vectors: $(\bar{y}_a \cdot y_b)$

$$\int F(\bar{y} \cdot y)\, dy\, d\bar{y} = \int_{\mathrm{U}(n)} F(U)\, \mathrm{Det}^{-N}(U)\, dm(U)$$

for some Haar measure $dm(U)$

The Simplest Example

Let $F : C \to C$ be an entire function.

For a single anticommuting vector y consider $F(\overline{y} \cdot y)$.

$$\int F(\overline{y} \cdot y)\, dy\, d\overline{y}$$

$$= \frac{\partial^2}{\partial y_1 \partial \overline{y}_1} \cdots \frac{\partial^2}{\partial y_N \partial \overline{y}_N} F(\overline{y}_1 y_1 + \ldots + \overline{y}_N y_N)$$

$= F^{(N)}(0)$ (the N-th derivative at the origin)

$$= \oint F(\mathrm{e}^{\mathrm{i}q})\, \mathrm{e}^{-\mathrm{i}Nq}\, dq / 2p \,.$$

Sketch of proof --- general case

Let $F : \mathrm{End}(C^n) \to C$ be an entire function.

For $U_L, U_R \in \mathrm{U}(n)$ define $F_{U_L,U_R}(M) := F(U_L M U_R)$.

Distribution 1: $m_1[F] = \int_{\mathrm{U}(n)} F(U)\, \mathrm{Det}^{-N}(U)\, dm(U)$

Distribution 2: $m_2[F] = \int F(\overline{y} \cdot y) \prod_{i,a} dy_i^a\, d\overline{y}_i^a$

The transformation behavior is the same:

$$m_A[F_{U_L,U_R}] = \mathrm{Det}^N(U_L)\, \mathrm{Det}^N(U_R)\, m_A[F] \qquad (A = 1,2)$$

$m_1 \propto m_2$ now follows since the action of the dual pair $\mathrm{U}(N) \times \mathrm{U}(2n)$ on the spinor representation of $\mathrm{Spin}(4nN)$ is multiplicity-free.

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for some Haar measure $dm(U)$

Supersymmetry Method

$$\mathrm{Det}^{-1}(E \pm \mathrm{i}e - H) = \int \mathrm{e}^{\pm \mathrm{i}(\bar{j},(E \pm \mathrm{i}e - H)j)}$$

$$\mathrm{Det}(E - H) = \int \mathrm{e}^{-\mathrm{i}(\bar{y},(E-H)y)}$$

The GUE_N average
$$\left\langle \mathrm{e}^{\pm \mathrm{i}(\bar{j},Hj) - \mathrm{i}(\bar{y},Hy)} \right\rangle_{\mathrm{GUE}_N}$$

depends only on the scalar products

$$M = \begin{pmatrix} (\bar{j},j) & (\bar{j},y) \\ (\bar{y},j) & (\bar{y},y) \end{pmatrix}.$$

Switch to the entries of the supermatrix M as the new variables of integration!

Superbosonization

Arrange scalar products $\bar{j}\cdot j\ ,\ \bar{j}\cdot y\ ,\ \bar{y}\cdot j\ ,\ \bar{y}\cdot y$

into a supermatrix M.

$$\int F\begin{pmatrix} \bar{j}\cdot j & \bar{j}\cdot y \\ \bar{y}\cdot j & \bar{y}\cdot y \end{pmatrix} dy\ d\bar{y}\ dj\ d\bar{j}$$

$$= \int DM\,\mathrm{SDet}^{N}(M)\,F(M)$$

Superdeterminant:

$$\mathrm{SDet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}(A-BD^{-1}C)}{\mathrm{Det}(D)} = \frac{\mathrm{Det}(A)}{\mathrm{Det}(D-CA^{-1}B)}$$

Supertrace: $\mathrm{STr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Tr}(A)-\mathrm{Tr}(D)$

The Simplest Example

$$p = \bar{j} \cdot j \;,\;\; x = \bar{j} \cdot y \;,\;\; h = \bar{y} \cdot j \;,\;\; q = \bar{y} \cdot y$$

$$F\begin{pmatrix} p & x \\ h & q \end{pmatrix} = (F_0 + x\,F_1 + h\,F_2 + xh\,F_3)(p,q)$$

$$\mathrm{SDet}^N \begin{pmatrix} p & x \\ h & q \end{pmatrix} = \frac{p^N}{q^N} - Nxh\,\frac{p^{N-1}}{q^{N+1}} \;.$$

$$\int F\begin{pmatrix} \bar{j} \cdot j & \bar{j} \cdot y \\ \bar{y} \cdot j & \bar{y} \cdot y \end{pmatrix} dy\; d\bar{y}\;\; dj\; d\bar{j}\;\; \times p^{\,N}$$

$$= \frac{1}{(N-1)!}\int_0^\infty (\partial_q^N F_0(p,q) - p\partial_q^{N-1} F_3(p,q))_{q=0}\; p^{N-1} dp$$

$$= \int DM\,\mathrm{SDet}^N(M)\,F(M) \quad \text{if} \quad DM = dp\,dq\,\partial_x\partial_h \,/\, 2p\,\mathrm{i}$$

A Simple Application

$\mathrm{U}(N)$-invariant probability density for Hermitian $N \times N$ matrices H:

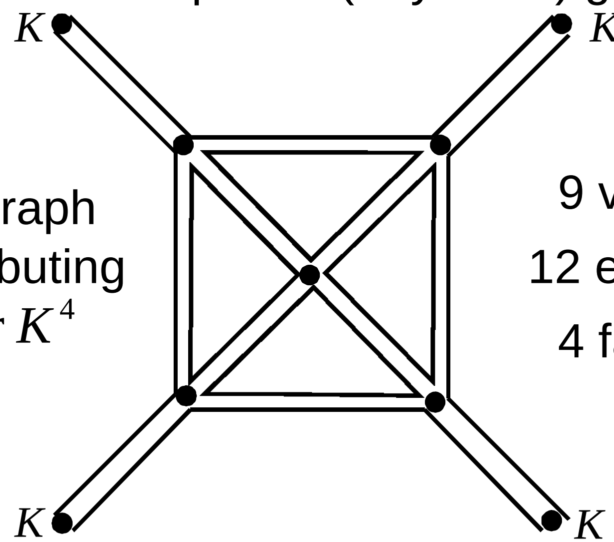
$$dm_{V,N}(H) = \mathrm{e}^{-N \,\mathrm{Tr}\, V(H)} dH \,.$$

Fourier transform (for large N):

$$\int \mathrm{e}^{-N \,\mathrm{Tr}\, V(H) + \mathrm{i} N \,\mathrm{Tr}\, HK} \, dH = \mathrm{e}^{-N \,\mathrm{Tr}\, \tilde{V}(K) + O(N^0)} .$$

$\mathrm{Tr}\, \tilde{V}(K)$ is a sum over planar (Feynman) graphs.

Ex.: graph contributing to $\mathrm{Tr}\, K^4$



9 vertices: N^9

12 edges: N^{-12}

4 faces: N^4

Every planar graph has Euler characteristic $c = 1$.

A Simple Application: Average Resolvent

Assume $\operatorname{Im} z > 0.$ Then: $\left\langle \frac{\mathrm{i}}{N} \operatorname{Tr}(z-H)^{-1} \right\rangle_m$

$$= \int \bar{j} \cdot j \left\langle \mathrm{e}^{\mathrm{i}N(\bar{j},(z-H)j) - \mathrm{i}N(\bar{y},(z-H)y)} \right\rangle_m$$

$(K = j \otimes \bar{j} + y \otimes \bar{y})$

$$= \int \bar{j} \cdot j \ \mathrm{e}^{\mathrm{i}Nz(\bar{j}\cdot j - \bar{y}\cdot y)} \left\langle \mathrm{e}^{-\mathrm{i}N \operatorname{Tr} HK} \right\rangle_m$$

$$= \int DM \, M_{jj} \ \mathrm{e}^{N \operatorname{STr}(\mathrm{i}zM + \ln M - \tilde{V}(M)) + O(N^0)}.$$

Saddle-point equation: $\boxed{\mathrm{i}z + M^{-1} = \tilde{V}'(M)}$

Universal Construction of Symmetric Superspaces

Complex Lie superalgebra $\mathbf{g} = \mathbf{g_0} + \mathbf{g_1}$ (Z_2-grading)

(with Cartan involution) $= (\mathbf{h_0} + \mathbf{p_0}) + (\mathbf{h_1} + \mathbf{p_1})$

Pick real Lie groups $H_0 \subset G_0$ such that $G_0 / H_0 \equiv X$

is globally symmetric Riemannian manifold.

H_0 acts on $\mathbf{p}_1$ by Ad.

Form associated vector bundle $E = G_0 \times_{H_0} \mathbf{p}_1 \to X$

$\mathbf{g}$ canonically acts on 'superfunctions' $\Gamma(X, \wedge E^*)$

Invariant Berezin form $\Gamma(X, \wedge E^*) \xrightarrow{\int_F} \Gamma(X, \wedge^{\text{top}} T^* X)$

Superbosonization

Propn: In the stable range there exists a Berezin form

$$DM : \Gamma(X, \wedge E^*) \to \Gamma(X, \wedge^{\text{top}} T^* X)$$ such that

$$\int dj\, d\bar{j} \int_{\mathrm{F}} \partial_y \partial_{\bar{y}}\, F(M(\bar{j}, j; \bar{y}, y))$$

$$= \int_X DM\ \mathrm{SDet}^N(M) F(M).$$

Proof: Both sides are distributions (i.e. continuous linear functionals) on $\Gamma(X, \wedge E^*)$. They transform in the same way under the action of the Lie superalgebra **g** .