Some properties of Wishart processes

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Some properties of Wishart processes

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Wishart processes

Wishart matrices

squared Bessel processes

(1) squared Bessel processes

$n \in \mathbb{N}^*$ $\quad BESQ(n) =$ law of $(X_t = \|B_t\|^2, t \geq 0)$

B_t $MB(\mathbb{R}^n)$

$$dX_t = 2\sqrt{X_t}\, d\beta_t + n\, dt$$

<u>Definition</u> $BESQ_x(\delta)$ unique strong solution of

$$\begin{cases} dX_t = 2\sqrt{X_t}\, d\beta_t + \delta\, dt & \delta \geq 0 \\ X_0 = x \geq 0 \end{cases}$$

Q_x^δ distribution on $C(\mathbb{R}^+, \mathbb{R}^+)$

Properties:

1) additivity property

$$Q_x^{\delta} * Q_y^{\delta'} = Q_{x+y}^{\delta+\delta'}$$

$\rightarrow (Q_x^{\delta})$ infinitely divisible on $C(\mathbb{R}^+, \mathbb{R}^+)$

2) local absolute continuity property

$$Q_x^{2\nu+2}\Big/\mathcal{G}_t^{\rho} = \left(\frac{X_t}{x}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s}\right) Q_x^{2}\Big/\mathcal{G}_t^{\rho}$$

$x > 0$

$\longrightarrow$

$$Q_x^{2}\left[\exp -\frac{\nu^2}{2}\int_0^t \frac{ds}{X_s} \Big/ X_t = y\right] = \frac{I_\nu(r)}{I_0(r)}$$

$$= \int_0^\infty e^{-\frac{\nu^2}{2}x}\, \eta_r(dx)$$

$r = \frac{\sqrt{xy}}{t}$

$\downarrow$

Hartman - Watson distribution

Applications:

- study of planar B.M.
- mathematical finance (Asian Options)

② Wishart matrices

Multivariate statistical theory (Muirhead)

$G = (G_{ij})$ $n \times m$ matrix G_{ij} $N(0,1)$ iid

$X = G'G \in \mathcal{S}_m^+$ = symmetric positive $m \times m$ matrices

$W_m(n)$: law of X on $\mathcal{S}_m^+$

$n \geq m$ $W_m(n)$ has a density

$$C_{n,m} \exp\left(-\frac{1}{2} \operatorname{Tr} y\right) \left(\det y\right)^{\frac{n-m-1}{2}} dy$$

We can extend $W_m(\delta)$ for

$$\delta \in \{1, 2.., m-1\} \cup]m-1, \infty[$$

Additivity:

$$W_m(\delta) * W_m(\delta') = W_m(\delta + \delta')$$

no infinitely divisible

Wishart processes

$\delta = n \qquad X_t = B_t' B_t \in \mathcal{S}_m^+$

$B_t = (B_{ij}(t)) \quad n \times m$ matrix

B_{ij} iid Brownian motions

$$\boxed{dX_t = \sqrt{X_t}\, dW_t + dW_t' \sqrt{X_t} + \delta I_m\, dt} \quad (1)$$

$W \quad m \times m$ Brownian motion

Theorem M.F. Bru (1989-1991)

$X_0 \in \mathcal{S}_m^+ \qquad \lambda_1(0) > \dots > \lambda_m(0) \geq 0$

(1) $\delta \in (m-1, m+1)$, (1) has a unique solution in law

(2) $\delta \geq m+1$ unique strong solution in $\tilde{\mathcal{S}}_m^+$ (strictly positive)

(3) The e.v. $(\lambda_i(t))$ satisfy :

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}\, d\beta_i(t) + \left\{ \delta + \sum_{j \neq i} \frac{\lambda_i(t) + \lambda_j(t)}{\lambda_i(t) - \lambda_j(t)} \right\} dt$$

$i = 1, \dots, m$

$\lambda_1(t) > \dots > \lambda_p(t)$

(4)

$$Q_x^\delta \left[\exp - \mathrm{Tr}(\Theta X_t) \right] = \det(I + 2t\Theta)^{-\delta/2} \exp\left[-\mathrm{Tr}\left(x (I + 2t\Theta)^{-1} \Theta \right) \right]$$

Properties:

1) $$q_t^\delta(x, dy) = \frac{1}{(2t)^{\delta m/2}\,\Gamma_m\left(\frac{\delta}{2}\right)} \exp\left(-\frac{1}{2t}\mathrm{Tr}(x+y)\right)(\det y)^{\frac{\delta-m-1}{2}}$$

$$\times\ {}_0F_1\left(\frac{\delta}{2}; \frac{xy}{4t^2}\right)$$

${}_0F_1$ hypergeometric function with matrix argument or Bessel function (Herz 55)

non central Wishart distribution

2) $\delta \geq m+1$
we can start from a degenerate initial condition $(x=0)$, $X_t \in \tilde{S}_t^+$ for $t>0$.

3) additivity: Q_x^δ distribution on $C(\mathbb{R}^+; S_p^+)$

$\delta \geq m-1$
$\delta' \geq m-1$

$$Q_x^\delta * Q_y^\delta = Q_{x+y}^{\delta+\delta'}$$

Absolute continuity relation

$\delta \geq m+1$

Girsanov's formula

W $m \times m$ Brownian motion driving the SDE (1)

$$\mathcal{E}_t^H = \exp\left(\int_0^t \mathrm{Tr}(H_s \, dW_s) - \frac{1}{2} \int_0^t \mathrm{Tr}(H_s^2) \, ds \right)$$

H_s predictable process with values in $\mathcal{S}_m$

$$Q_x^{\delta, H} / \mathcal{F}_t^\rho = \mathcal{E}_t^H \; Q_x^\delta / \mathcal{F}_t^\rho \qquad x \in \widetilde{\mathcal{S}}_m^+$$

Then,

$$W_t = B_t + \int_0^t H_s \, ds$$

$Q_x^{\delta, H}$ B.M.

Under $\mathbb{Q}_x^{\delta, H}$

$$\boxed{dX_t = \sqrt{X_t}\, dB_t + dB_t' \sqrt{X_t} + \left(\sqrt{X_t}\, H_t + H_t \sqrt{X_t} + \delta I_m\right) dt}$$

Take $H_t = \nu X_t^{-1/2}$

Proposition: $\delta = m+1+2\nu \leftarrow$ index

$$\mathbb{Q}_x^{(\nu)}\Big/\mathcal{G}_t^{\rho} = \left(\frac{\det X_t}{\det x}\right)^{\nu/2} \exp\left(-\frac{\nu^2}{2}\int_0^t \mathrm{Tr}\left(X_s^{-1}\right) ds\right) \mathbb{Q}_x^{(0)}\Big/\mathcal{G}_t^{\rho}$$

Corollary

$$\mathbb{Q}_x^{(0)}\left[\exp -\frac{\nu^2}{2}\int_0^t \mathrm{Tr}\left(X_s^{-1}\right) ds \Big/ X_t = y\right]$$

$$= C_{m,\nu} \left(\det z\right)^{\nu/2} \frac{{}_0F_1\left(\frac{m+1}{2}+\nu\,;\, z\right)}{{}_0F_1\left(\frac{m+1}{2}\,;\, z\right)}$$

$$z = \frac{xy}{4t^2} \qquad C_{m,\nu} = \frac{1}{\Gamma_m\left((m+1)/2+\nu\right)}$$

Time inversion:

$$X_t \quad WIS(\delta; x)$$

$$Y_t := t^2 X_{1/t}$$

$x = 0$ then $Y \sim WIS(\delta; 0)$

Definition: Wishart process with drift

$\Delta =$ drift $\in \mathcal{S}_m^+$

$WIS^{(\Delta)}(\delta; x)$ Markov process with S.G.

$$q^{\delta,(\Delta)}(t, x, dy) = \frac{{}_0F_1(\delta/2; \Delta y/4)}{{}_0F_1(\delta/2; \Delta x/4)} e^{-\frac{1}{2}\mathrm{Tr}(\Delta)t} q^{\delta}(t, x, dy)$$

$$x, y \in \mathcal{S}_m^+$$

For $\delta = m$, law of $(B_t + \theta t)'(B_t + \theta t)$

B_t, θ $m \times m$ $\qquad \Delta = \theta'\theta$

Proposition:

X $WIS(\delta, x) \longrightarrow Y$ $WIS^{(x)}(\delta; 0)$

X $WIS^{(\Delta)}(\delta; x) \longrightarrow Y$ $WIS^{(x)}(\delta; \Delta)$

Intertwining relation

$$X \sim W_m(\delta)$$
$$Y \sim W_m(\delta')$$

independent

$$\left(X+Y,\ (X+Y)^{-1/2} X (X+Y)^{-1/2}\right) \overset{\text{law}}{=} \left(W_m(\delta+\delta'),\ \mathrm{Beta}_m\left(\frac{\delta}{2},\frac{\delta'}{2}\right)\right)$$

independent

We have a Markov extension

<u>Proposition:</u> $f : \mathcal{S}_m^+ \longrightarrow \mathbb{R}$

$$\boxed{Q_t^{\delta+\delta'} \Lambda f(x) = \Lambda Q_t^{\delta} f(x)}$$

where Λ is a kernel given by

$$\Lambda f(x) = E\left[f(\sqrt{x}\, \beta \sqrt{x}) \right]$$

$$\beta \sim \mathrm{Beta}_m\left(\frac{\delta}{2}, \frac{\delta'}{2}\right)$$

Limiting results

Proposition: $X \quad WIS(\delta, x) \quad x \in \tilde{\mathcal{S}}_m^+$

1) $\delta = m+1$

$$\frac{4}{m^2(\log(t))^2} \int_0^t \mathrm{Tr}(X_u^{-1})\, du \xrightarrow[t\to\infty]{\text{law}} T_1(\beta)$$

$$T_1(\beta) = \inf\{t>0,\ \beta t = 1\} \qquad \beta \quad MB_0(\mathbb{R})$$

2) $\delta > m+1$

$$\frac{1}{\log t} \int_0^t \mathrm{Tr}(X_u^{-1})\, du \xrightarrow[t\to\infty]{} \frac{m}{\delta - m - 1} \quad \text{a.s.}$$

hypergeometric function

$$ {}_0F_1(a; X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(X)}{(a)_\kappa \, k!} $$

κ partition $\quad \kappa = (k_1, \dots, k_m) \quad k_1 \geq \dots \geq k_m$

$$ k = \sum_{i=1}^{m} k_i $$

$$ \kappa! = k_1! \dots k_m! $$

$$ (a)_\kappa = \prod_{i=1}^{m} \left(a - \frac{i-1}{2}\right)_{k_i} \qquad (a)_k = a(a+1)\dots(a+k-1) $$

$C_\kappa(X)$ zonal polynomial $\quad X \in \mathcal{S}_m$

(symmetric homogeneous in the e.v. of X)

Integral representation:

$X \in M_{m,n} \quad m \leq n \qquad H = (\overset{m}{\overbrace{H_1}}, H_2) \in O(n)$

$$ {}_0F_1\left(\frac{n}{2}, \frac{1}{4} \underbrace{XX'}_{m \times m}\right) = \int_{O(n)} \exp\left(\mathrm{Tr}(X H_1)\right) dH $$

characterisation by differential equations

${}_0F_1(c, Y)$ solution of

$$y_i \frac{\partial^2 F}{\partial y_i^2} + \left\{ c - \frac{1}{2}(m-1) + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{y_i}{y_i - y_j} \right\} \frac{\partial F}{\partial y_i}$$

$$- \frac{1}{2} \sum_{j \neq i} \frac{y_j}{y_i - y_j} \frac{\partial F}{\partial y_j} = F$$

$i = 1, \ldots, m$