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How long does it take to exit the chamber ?

Joint work with N. O'Connell

- **How long** before independent Brownian walkers **collide**?

- In mathematical terms, $C = \{x \in \mathbb{R}^n : \ x_1 > x_2 > \cdots > x_n\}$, $T =$ **exit time** from C for $\mathrm{BM}(\mathbb{R}^n)$.

- **Explicit** formula for $p = \mathbb{P}_x(T > t)$?

Some geometry

- $C = \{x \in \mathbb{R}^n : \; x_1 > x_2 > \cdots > x_n\}$ is a very special **cone**.

- **Walls :** hyperplanes $\{x \in \mathbb{R}^n : \; x_i = x_j\} = (e_i - e_j)^\perp$.

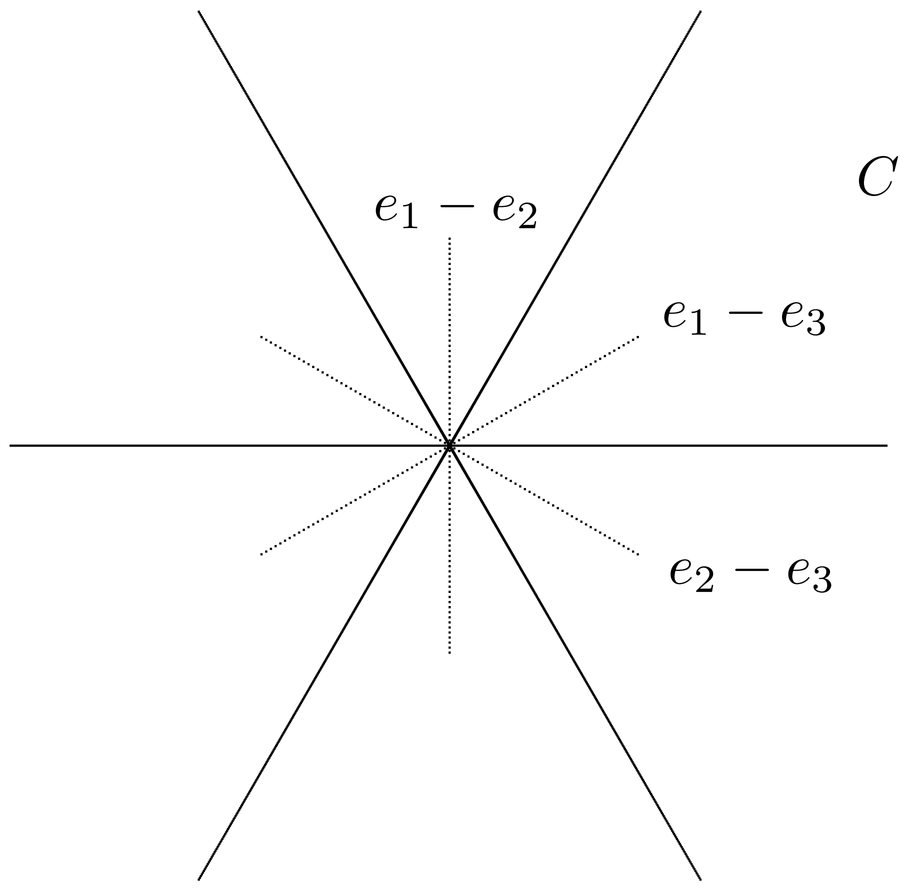
- **Roots :** $\Phi := \{e_i - e_j, \; i \neq j\}$.

- **Positive roots :** $\Phi^+ := \{e_i - e_j, \; i < j\}$ such that $C = \{x : \; (x, \alpha) > 0 \text{ for } \alpha \in \Phi^+\}$.

- **Simple roots :** $\Delta := \{e_i - e_{i+1}\}$, corresponding to hyperplanes **bordering** C. Then, $C = \{x : \; (x, \alpha) > 0 \text{ for } \alpha \in \Delta\}$.

EXAMPLE : $C = \{x_1 > x_2 > x_3\}$

$\Delta = \{e_1 - e_2, e_2 - e_3\}$, $\Phi^+ = \{e_1 - e_2, e_2 - e_3, e_1 - e_3\}$



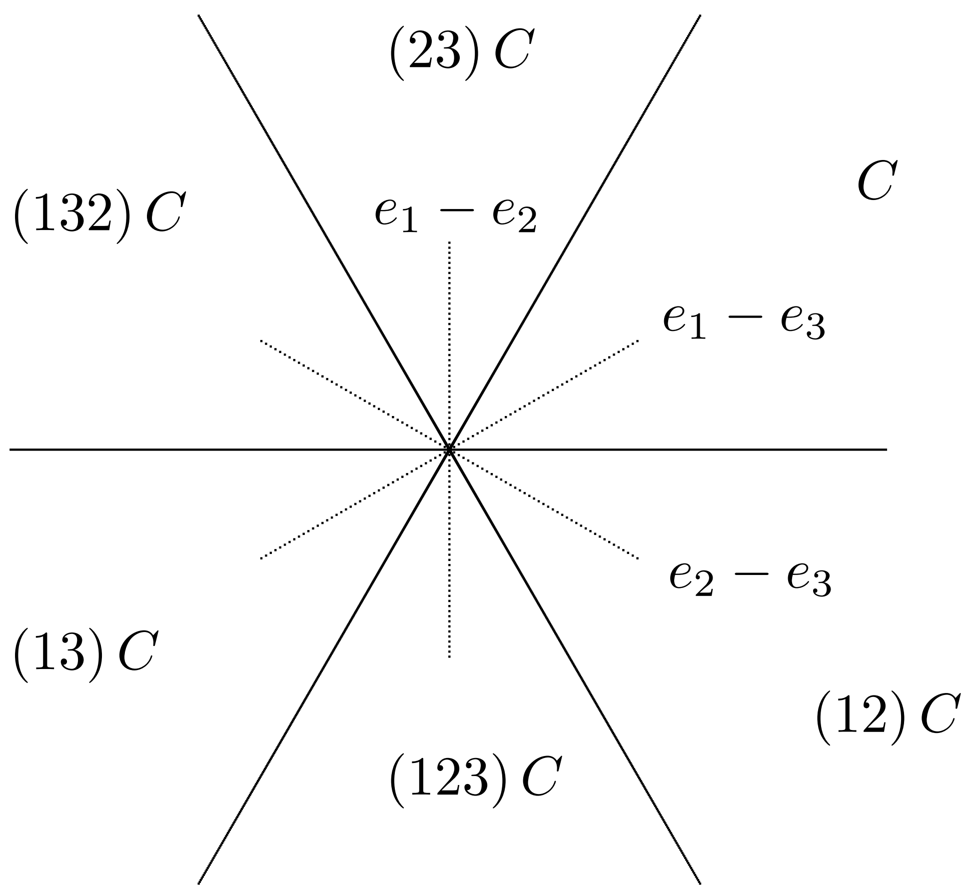
Some algebra

- $s_\alpha :=$ **reflection** with respect to $\alpha^\perp$.
  ($s_{e_i - e_j}$ corresponds with the transposition (ij)).

- $W := \langle s_\alpha,\ \alpha \in \Phi \rangle = \langle s_\alpha,\ \alpha \in \Delta \,|\, (s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \rangle$ finite **Coxeter** group.
  ($W = \langle (ij),\ i \neq j \rangle = \mathfrak{S}_n$).

- C is a **fundamental domain** for the action of W on $\mathbb{R}^n$ (the wC, $w \in W$, are the connected components, called **chambers**, of $\mathbb{R}^n \setminus \cup_{\alpha \in \Phi} \alpha^\perp$).

$C = \{x_1 > x_2 > x_3\}$
$W = \mathfrak{S}_3 = \langle\, (12),\, (23),\, (13)\, \rangle = \langle\, (12),\, (23)\, \rangle$



Main result

- For $I \subset \Phi^+$, define its positive orbit $\mathcal{I} := \{A = wI : \; wI \subset \Phi^+\}$.

- If **smart choice** of I, then $\epsilon_A := \det w$ for $A = wI \in \mathcal{I}$ is well-defined and the following holds :

$$\boxed{\mathbb{P}_x(T > t) = \sum_{A \in \mathcal{I}} \epsilon_A \, \mathbb{P}_x(T_A > t)} \, ,$$

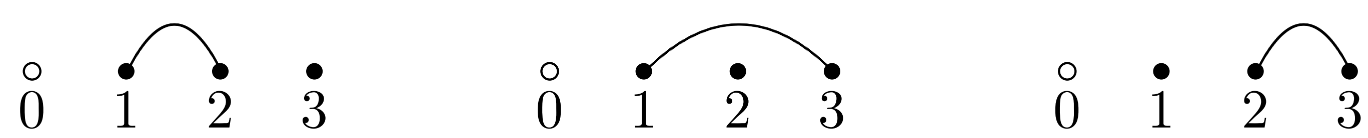
  where $T_A := \min_{\alpha \in A} T_\alpha$ and $T_\alpha :=$ hitting time of $\alpha^\perp$.

- If I is **orthogonal**, then $\mathbb{P}_x(T_A > t) = \prod_{\alpha \in A} \mathbb{P}_x(T_\alpha > t)$ (terms in the product are **one-dimensional** integrals).

Case $n = 3$

$$\left|\begin{matrix} I = \{e_1 - e_2\} \\ \epsilon = 1 \end{matrix}\right| \xrightarrow{(23)} \left|\begin{matrix} \{e_1 - e_3\} \\ \epsilon = -1 \end{matrix}\right| \xrightarrow{(12)} \left|\begin{matrix} \{e_2 - e_3\} \\ \epsilon = 1 \end{matrix}\right|$$

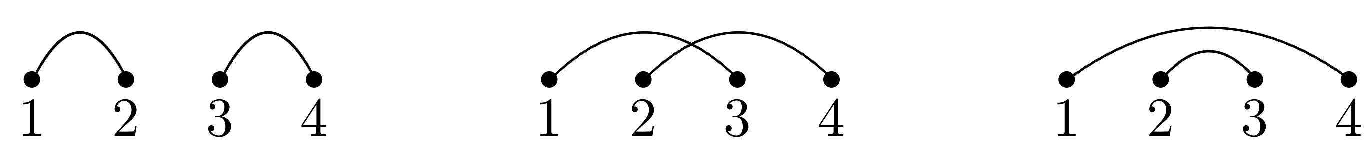
$$\boxed{p = p_{12} - p_{13} + p_{23}}$$



Case $n = 4$

$$\left| \begin{matrix} \{e_1 - e_2, e_3 - e_4\} \\ \epsilon = 1 \end{matrix} \right| \xrightarrow{(23)} \left| \begin{matrix} \{e_1 - e_3, e_2 - e_4\} \\ \epsilon = -1 \end{matrix} \right| \xrightarrow{(12)} \left| \begin{matrix} \{e_2 - e_3, e_1 - e_4\} \\ \epsilon = 1 \end{matrix} \right|$$

$$\boxed{p = p_{12}\, p_{34} - p_{13}\, p_{24} + p_{14}\, p_{23}}$$



General n

- In general, take $I = \{e_1 - e_2, e_3 - e_4, e_5 - e_6, \ldots\}$.
- Then, $\mathcal{I}$ identified with $P(n) := \{$ pair partitions of $\{1, \ldots, n\}\}$.
- The formula has the following form :
$$p = \sum_{\pi \in P(n)} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} p_{ij} := \mathrm{Pf}\,(p_{ij})\,,$$
where $c(\pi)$ = number of crossings of π.
- $\mathrm{Pf}\,^2(p_{ij}) = \det A$, where $A_{ij} = p_{ij} = -p_{ji}$ (n even).

Connection with the reflection principle

- Generalized reflection principle ([G-Z,B]) :

$$\mathbb{P}_x(B_t = y,\, T > t) = \sum_{w\in W} \epsilon(w)\,\mathbb{P}_x(B_t = wy) = \det\left(p_t(x_i, y_j)\right).$$

- By integration,

$$\mathbb{P}_x(T > t) = \sum_{w\in W} \epsilon(w)\,\mathbb{P}_x(B_t \in wC) = \int_C \det\left(p_t(x_i, y_j)\right)\, dy$$

.

- So we recover de Bruijn's formula :

$$\int_{y_1>\cdots>y_n} \det\left(f_i(y_j)\right)\, dy_1 \ldots dy_n = \mathrm{Pf}(\int \mathrm{sgn}(\lambda-\mu)\, f_i(\lambda) f_j(\mu)\, d\lambda d\mu)\,.$$

Is the formula useful ?

- Extends to many **other reflection groups** and associated domains (ex : $C = \{x_1 > \cdots > x_{n-1} > |x_n|\}$, wedges of angle π/m in $\mathbb{R}^2$, F_4, H_3, etc), giving generalized de Bruijn's formulae.

- Possible to derive **asymptotic expansions** of $\mathbb{P}_x(T > t)$ both for large and small time t (large-time ones are connected with **Selberg's integral**).

- Possible to compute **expected values** $\mathbb{E}_x(T)$ (giving solutions of Poisson equation).

- Discrete versions hold, allowing us to recover results by Gessel and Stembridge on the number of **Young tableaux** of bounded height.