# Asymptotics of Haar unitaries and their truncations

based on joint work with Dénes Petz

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## References

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**1. The group $\mathcal{U}(n)$:**

$\mathcal{U}(n)$ the topological group of $n \times n$ unitary matrices, $\gamma_n$, the Haar measure on $\mathcal{U}(n)$. $U_n$ is a Haar unitary random matrix, if $\mathcal{U}(n)$, and

$$\text{Prob}\,(U_n \in H) = \gamma(H).$$

**2. To get a Haar unitary:**

Let $\xi$ be a complex-valued random variable. If Re$\xi$ and Im$\xi$ are independent and normally distributed according to $N(0, 1/2)$, then we call $\xi$ a *standard complex normal variable*.

If $Z = (Z_{ij})$, where $Z_{ij}$ i.i.d. standard complex normal random variables. After the Gram-Schmidt ortogonalization procedure on the column vectors of $Z$, we get a Haar unitary random matrix.

## 2. Entries and eigenvalues:

From the construction the distribution of the entries:

$$\frac{n-1}{\pi}(1-r^2)^{n-2} r\, dr\, d\theta$$

($\sqrt{n}U_{ij}$ converges to a standard complex normal variable in distribution.)

The correlation coefficient $\rho$ if $i \neq j$

$$\rho(|U_{ii}|^2, |U_{jj}|^2) = \frac{1}{(n-1)^2},$$

and

$$\rho(|U_{ii}|^2, |U_{ij}|^2 = -\frac{1}{(n-1)}.$$

The eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, and their joint distribution is:

$$\frac{1}{n!}\prod_{i<j}|z_i - z_j|^2 = \frac{1}{n!}\prod_{i<j}|e^{i\theta_i} - e^{i\theta_j}|^2$$

with respect to $dz_0\, dz_1 \ldots dz_{n-1}$, where

$$dz_i = d\theta_i/2\pi$$

for $z_i = e^{i\theta_i}$.

## 3. Powers of a Haar unitary:

**Theorem 1** *For $m > n$ the random variables $\lambda_0^m, \lambda_1^m, \ldots, \lambda_{n-1}^m$ are independent and uniformly distributed on $\mathbb{T}$.*

Proof: We show that for $k_1, \ldots k_{m-1} \in \mathbb{Z}$

$$\int z_0^{k_0 m} z_1^{k_1 m} \ldots z_{n-1}^{k_{n-1} m} \prod_{i<j} |z_i - z_j|^2 \, dz = 0$$

if at least one $k_j \neq 0$ ($dz = dz_0 \, dz_1 \ldots dz_{n-1}$ and integration is over $\mathbb{T}^n$).

**Theorem 2** *Let $Z$ be standard complex normal random variable, and let $U_n$ be a sequence of Haar unitary random matrices. Then $\operatorname{Tr} U_n, \operatorname{Tr} U_n^2, \ldots, \operatorname{Tr} U_n^k$ are asymptotically independent and $\operatorname{Tr} U_n^k \to \sqrt{k} Z$ in distribution as $n \to \infty$.*

Proof: Check the convergence of the joint moments of $\operatorname{Tr} U_n, \operatorname{Tr} U_n^2, \ldots \operatorname{Tr} U_n^k$.

## 4. Truncation of a Haar unitary:

Let $U_n$ be an $n \times n$ Haar unitary random matrix. By truncating $n-m$ bottom rows and $n-m$ last columns, we get an $m \times m$ matrix $U_{[n,m]}$. The eigenvalues $z_1, z_2, \dots, z_m \in D^m$, where $D = \{z \in \mathbb{C} : |z| \le 1\}$ is the unit disc. According to *Życzkowski and Sommers* the probability density of the eigenvalues is

$$C_{[n,m]} \prod_{1 \le i < j \le m} |\zeta_i - \zeta_j|^2 \prod_{i=1}^{m} (1 - |\zeta_i|^2)^{n-m-1}$$

on $D^m$.

The normalizing constant was calculated by Petz and Réffy:

$$C_{[n,m]}^{-1} = \pi^m m! \prod_{k=0}^{m-1} \binom{n-m+k-1}{k}^{-1} \frac{1}{n-m+k}.$$

**Theorem 3** *The normalized truncated matrix*

$$\sqrt{\frac{n}{m}} U_{[n,m]}$$

*converge in distribution to the standard* $m \times m$ *non-selfadjoint Gaussian matrix as* $n \to \infty$*.*

Proof: Compute the limit of the joint eigenvalue distribution.

**5. Introduction to large deviations:** Let $(P_n)$ be a sequence of measures on a topological space $\mathcal{X}$. The large deviation principle holds with rate function $I$ in the scale $L(n)$ if

$$\liminf_n L(n) \log P_n(G) \geq -\inf\{I(x) : x \in G\}$$

for every open set $G \subset \mathcal{X}$ and

$$\limsup_n L(n) \log P_n(F) \leq -\inf\{I(x) : x \in F\}$$

for every closed set $F \subset \mathcal{X}$, where

$$L(n) \leq C/n.$$

(If the latter condition holds only for compact sets $F$, then the weak large deviation principle is said to hold true.) Here $I : \mathcal{X} \to [0, \infty]$ is lower semicontinuous and is called a good rate function if $\{x \in \mathcal{X} : I(x) \leq c\}$ is compact for every $c \geq 0$.

## 6. LDP for random matrices:

Assume that $T_n(\omega)$ is a random $n \times n$ matrix with complex eigenvalues $\zeta_1(\omega), \ldots, \zeta_n(\omega)$. The *empirical eigenvalue density* of $T_n(\omega)$ is

$$P_n(\omega) := \frac{\delta(\zeta_1(\omega)) + \cdots + \delta(\zeta_n(\omega))}{n},$$

where $\delta(z)$ denotes the Dirac measure supported on $\{z\} \subset \mathbb{C}$. Therefore $P_n$ is a measure on the set of measures.

The first large deviation theorem for random matrices was obtained by *Ben Arous and Guionnet* in 1997 and it concerns the empirical eigenvalue density of Gaussian symmetric (self-adjoint) random matrices as the matrix size tends to infinity.

## 7. LDP for random unitaries:

Let $Q(\zeta)$ be a real continuous function on $\mathbb{T}$ and for each $n \in \mathbf{N}$ set a probability measure $\nu_n$ on $\mathcal{U}(n)$ as

$$\nu_n = \frac{1}{Z_n} \exp(-n \mathrm{Tr}\, Q(U))\, d\gamma_n(U)\,,$$

where $Z_n$ is for normalization. We have the following large deviation theorem for the sequence of random unitaries distributed according to $\nu_n$.

**Theorem 4** *Let $P_n$ ($n \in \mathbf{N}$) be the probability measures defined above on $\mathcal{M}(\mathbb{T})$. Then the finite limit $B = \lim_{n\to\infty} n^{-2} \log Z_n$ exists and $(P_n)$ satisfies the large deviation principle in the scale $n^{-2}$ with rate function*

$$I(\mu) = \\ -\iint_{\mathbb{T}^2} \log|\zeta - \eta|\, d\mu(\zeta)\, d\mu(\eta) + \int_{\mathbb{T}} Q(\zeta)\, d\mu(\zeta) + B$$

*for $\mu \in \mathcal{M}(\mathbb{T})$. Furthermore, there exists a unique $\mu_0 \in \mathcal{M}(\mathbb{T})$ such that $I(\mu_0) = 0$.*

**Example:** Let $\lambda > 0$ and $Q(\zeta) = -\frac{2}{\lambda}\mathrm{Re}\zeta$ ($\zeta \in \mathbb{T}$). Then $\nu_n$ on $\mathcal{U}(n)$ is

$$\nu_n = \frac{1}{Z_n}\exp\left(\frac{n}{\lambda}\mathrm{Tr}\,(U+U^*)\right)d\gamma_n(U)\,,$$

and the rate function is

$$-\iint_{\mathbb{T}^2}\log|\zeta-\eta|\,d\mu(\zeta)\,d\mu(\eta)-\frac{2}{\lambda}\int\mathrm{Re}\zeta\,d\mu(\zeta)+B_\lambda$$

for $\mu \in \mathcal{M}(\mathbb{T})$

Gross and Witten showed that the unique minimizer $\rho_\lambda$ of $I$ is given by

$$\frac{1}{2\pi}\left(1+\frac{2}{\lambda}\cos\theta\right)d\theta$$

if $\lambda \geq 2$ and

$$\frac{2}{\pi\lambda}\cos\frac{\theta}{2}\sqrt{\frac{\lambda}{2}-\sin^2\frac{\theta}{2}}\,d\theta$$

on the interval $[-2\arcsin\sqrt{\lambda/2},\,2\arcsin\sqrt{\lambda/2}\,]$ if $0 < \lambda < 2$.

## 7. LDP for truncated unitaries:

**Theorem 5** *Let $U_{[m,n]}$ be the $n \times n$ truncation of an $m \times m$ Haar unitary random matrix and let $1 < \lambda < \infty$. If $m/n \to \lambda$ as $n \to \infty$, then the sequence of empirical eigenvalue densities $P_n = P_{[m,n]}$ satisfies the large deviation principle in the scale $1/n^2$ with rate function*

$$I(\mu) = -\int\int_{\mathcal{D}^2} \log |z-w| \, d\mu(z) \, d\mu(w)$$

$$-(\lambda - 1) \int_{\mathcal{D}} \log(1-|z|^2) \, d\mu(z) + B,$$

*for $\mu \in \mathcal{M}(\mathcal{D})$, where $B =$*

$$-\frac{\lambda^2 \log \lambda}{2} + \frac{\lambda^2 \log(\lambda-1)}{2} - \frac{\log(\lambda-1)}{2} + \frac{\lambda-1}{2}.$$

*Furthermore, there exists a unique $\mu_0 \in \mathcal{M}(\mathcal{D})$ given by the density*

$$d\mu_0(z) = \frac{(\lambda-1)r}{\pi\left(1-r^2\right)^2} \, dr \, d\varphi, \qquad z = re^{\mathrm{i}\varphi}$$

*on $\{z : |z| \le 1/\sqrt{\lambda}\}$ such that $I(\mu_0) = 0$.*

## 7. Method of proof:

### A: Limit of the normalizing constant $C_{[m,n]}$:

$$
\begin{aligned}
B =: & \lim_{n\to\infty} \frac{1}{n^2} \log C_{[m,n]} \\
&= - \lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n-1} \log \binom{m-n+j-1}{j} \\
&= - \lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n-1} \sum_{i=1}^{j} \log \frac{m-n-1+i}{i} \\
&= - \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n-1} (n-1-i) \log \frac{m-n-1+i}{i} \\
&= - \lim_{n\to\infty} \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{n-1-i}{n-1} \log \frac{m-n-1+i}{i}.
\end{aligned}
$$

The limit of a Riemannian sum gives an integral:

$$
\int_0^1 (1-x) \log\left(\frac{\lambda-1+x}{x}\right) dx
$$

**B: The upper estimate:**

$$\inf_G \left\{ \limsup_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\}$$

$$\leq - \iint_{\mathcal{D}^2} F(z,w)\, d\mu(z)\, d\mu(w) - B$$

where $G$ runs over a neighbourhood base of $\mu$.

**C: The lower estimate:**

For every $\mu \in \mathcal{M}(\mathcal{D})$,

$$\inf_G \left\{ \liminf_{n\to\infty} \frac{1}{n^2} \log P_n(G) \right\}$$

$$\geq - \iint_{\mathcal{D}^2} F(z,w)\, d\mu(z)\, d\mu(w) - B,$$

where $G$ runs over a neighbourhood base of $\mu$.

By regularization of $\mu$.

## Minimizing the rate function

**Theorem 6** *Let* $f : \mathcal{D} \to (-\infty, \infty]$, *such that* $f(z) = f(|z|)$, $\forall z \in \mathcal{D}$, $f \in \mathcal{C}^1((0,1))$, $rf'(r)$ *increasing on* $(0,1)$

$$\lim_{r \to 1} rf'(r) = \infty.$$

*Set* $r_0 \geq 0$ *the smallest number such that* $f'(r) > 0$, $\forall r > r_0$, *and set* $R_0$ *the smallest number such that* $R_0 f'(R_0) = 1$.
*Clearly* $0 \leq r_0 < R_0 < 1$. *Set*

$$I_f := \int\int_{\mathcal{D}^2} \log \frac{1}{|z-w|} \, d\mu(z) \, d\mu(w) + 2 \int_{\mathcal{D}} f(z) \, d\mu(z)$$

*is minimal at* $\mu_f$ *on*

$$S_f = \{z : r_0 \leq |z| \leq R_0\},$$

*and the density of* $\mu_f$ *is*

$$d\mu_f(z) = \frac{1}{2\pi} (rf'(r))' \, dr \, d\varphi, \qquad z = re^{i\varphi}.$$

Now we use the above theorem for

$$f(z) := -\frac{\lambda - 1}{2} \log \left(1 - |z|^2\right).$$

## 8. Some connection to free probability:

Let $Q_m$ be an $m \times m$ projection matrix of rank $n$, and let $U_m$ be an $m \times m$ Haar unitary. Then the matrix $Q_m U_m Q_m$ has the same non-zero eigenvalues as $U_{[m,n]}$, but it has $m - n$ zero eigenvalues. The large deviation result for $U_{[m,n]}$ is easily modified.

Now let $\mathcal{M}$ be a von Neumann algebra and $\tau$ be a faithful normal trace on $\mathcal{M}$. The pair $(\mathcal{M}, \tau)$ is often called a *non-commutative probability space*. A unitary $u \in \mathcal{M}$ is called a *Haar unitary* if $\tau(u^k) = 0$ for every non-zero integer $k$. Let $q \in \mathcal{M}$ be a projection such that $\tau(q) = \lambda^{-1}$. If $u$ and $q$ are *free*, then the above $(U_m, Q_m)$ is random matrix model of the pair $(u, q)$. This means that

$$\frac{1}{m} E \left( \mathrm{Tr}\, \mathcal{P}(U_m, U_m^*, Q_m) \to \tau \left( \mathcal{P}(u, u^*, q) \right) \right.$$

for any polynomial $\mathcal{P}$ of three non-commuting indeterminants. This statement is a particular case of Voiculescu's fundamental result about *asymptotic freeness*.

For an element $a$ of the von Neumann algebra $\mathcal{M}$, the *Fuglede-Kadison* determinant can be defined by:

$$\Delta(a) := \lim_{\varepsilon \to +0} \exp \tau \left( \log(a^* a + \varepsilon I)^{1/2} \right).$$

It was shown by L.G. Brown in 1983 that the function

$$z \mapsto \frac{1}{2\pi} \log \Delta(a - zI)$$

is subharmonic and its Laplacian (taken in the distribution sense) is a probability measure $\mu_a$ concentrated on the spectrum of $a$. This measure is called the *Brown measure* and it is a sort of extension of the spectral multiplicity measure of normal operators:

$$\tau(g(a)) = \int_{\mathbb{C}} g(z)\, d\mu_a(z)$$

for any function $g$ on $\mathbb{C}$ that is analytic in a domain containing the spectrum of $a$.

Let $u$ be a Haar unitary, and $q = q^* = q^2$ be free from $u$. Then $uq$ is a so-called *R-diagonal* operator and its Brown measure is rotation invariant in the complex plane. The Brown measure has an atom of mass $1-\lambda^{-1}$ at zero and the absolute continuous part has a density

$$\frac{(\lambda-1)r}{\pi\lambda\left(1-r^2\right)^2}\,dr\,d\varphi \qquad (z = re^{\mathrm{i}\varphi})$$

on $\{z : |z| \leq 1/\sqrt{\lambda}\}$. We just observe that this measure coincides with the limiting measure in our large deviation theorem. In the moment we do not have a good explanation for this coincidence but it is definitely worthwhile to study it.