Long time propagation of coherent states under perturbed cat dynamics

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INTRODUCTION

QUANTUM CHAOS : the semi-classical analysis of quantum systems having a chaotic Hamiltonian system as their classical limit.

EXAMPLES:

1. *Quantum* : Δ on $L^2(M, \mathrm{dvol}(g))$; $\Delta\psi_n = \lambda_n \psi_n$;
   Chaotic system : the geodesic flow on (M, g), a compact negatively curved manifold.
   Semi-classical : $\lambda_n \to +\infty$.
2. *Quantum* : the Dirichlet laplacian Δ_{D} on a domain $\Omega \subset \mathbb{R}^2$; $\Delta\psi_n = \lambda_n \psi_n$
   Chaotic system : the billiard flow on Ω (Bunimovich, Sinai).
   Semi-classical : $\lambda_n \to +\infty$.
3. *Quantum* : quantum maps $=$ unitary maps on N dimensional spaces.
   Chaotic system : symplectic Anosov maps on the torus.
   Semi-classical : $N \to +\infty$.

ANOSOV MAPS ON THE TORUS

Hyperbolic automorphisms : $A \in \mathrm{SL}(2, \mathbb{Z})$, $| \mathrm{Tr} A | > 2 \;\Rightarrow\; Av_\pm = \mathrm{e}^{\pm\gamma_0} v_\pm$.

A acts as a symplectomorphism on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and is

- Hyperbolic : a.e. $x, x' \in \mathbb{T}^2$, $t \in \mathbb{N}$ (not too large),

$$\mathrm{d}(x, x') \sim \epsilon \Rightarrow \mathrm{d}(A^t x, A^t x') \sim \epsilon \mathrm{e}^{\gamma_0 t}.$$

- Exponentially mixing : $\forall f, g \in C^\infty(\mathbb{T}^2)$,

$$\left| \int_{\mathbb{T}^2} (f \circ A^t)(x) g(x) dx - \int_{\mathbb{T}^2} f(x) dx \int_{\mathbb{T}^2} g(x) dx \right| \leq C_{A,f} \parallel \nabla g \parallel_1 \; \mathrm{e}^{-\gamma_0 t}.$$

Perturbed hyperbolic automorphisms : For $g \in C^\infty(\mathbb{T}^2)$, $\phi_s, s \in \mathbb{R}$ is the Hamiltonian flow of g. Define

$$\Phi_\epsilon = \phi_\epsilon \circ A.$$

For small ϵ, this is still hyperbolic and exponentially mixing, with exponent γ_ϵ, $\lim_{\epsilon \to 0} \gamma_\epsilon = \gamma_0$ (Blank, Keller, Liverani 2002).

THE CORRESPONDING QUANTUM MAP

The Hilbert spaces With $U(a)\psi(y) = \mathrm{e}^{-\frac{i}{2\hbar}a_1 a_2}\mathrm{e}^{\frac{i}{\hbar}a_2 y}\psi(y - a_1) = \mathrm{e}^{-\frac{i}{\hbar}(a_1 P - a_2 Q)}\psi(y)$, define

$$\mathcal{H}_\hbar = \{\psi \in \mathcal{S}'(\mathbb{R}) \mid U(1,0)\psi = \psi = U(0,1)\psi\}, \quad 2\pi\hbar N = 1 \Rightarrow \mathrm{dim}\mathcal{H}_\hbar = N.$$

Then

$$\psi \in \mathcal{H}_\hbar \Rightarrow \psi(y) = \sum_{\ell\in\mathbb{Z}} c_\ell \delta(y - \frac{\ell}{N}); \quad c_{\ell+N} = c_\ell.$$

Weyl quantization For $f \in C^\infty(\mathbb{T}^2)$, $x = (q,p) \in \mathbb{T}^2$, write

$$f(x) = \sum_{n\in\mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 p - n_2 q)}$$

and define

$$\mathrm{Op}^{\mathrm{W}} f = \hat{f} = \sum_{n\in\mathbb{Z}^2} f_n \mathrm{e}^{-i2\pi(n_1 P - n_2 Q)} = \sum_{n\in\mathbb{Z}^2} f_n U(\frac{n}{N}) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar.$$

The quantum dynamics Take $A \in \mathrm{SL}(2,\mathbb{Z})$, $|\,\mathrm{Tr} A\,| > 2$ and construct

$$M(A)\psi(y) = \left(\frac{i}{2\pi\hbar a_{12}}\right)^{1/2} \int_{\mathbb{R}} \mathrm{e}^{\frac{i}{2\hbar a_{12}}(a_{22}y^2 - yy' + a_{11}y'^2)}\psi(y')dy'.$$

Then, for all $t \in \mathbb{Z}$,

$$M(A)\mathcal{H}_\hbar = \mathcal{H}_\hbar \quad \text{and} \quad M(A)^{-t}\,\mathrm{Op}^{\mathrm{W}} f\, M(A)^t - \mathrm{Op}^{\mathrm{W}}(f \circ A^t) = 0.$$

Now, for $\epsilon > 0$ define the unitary operator

$$U_\epsilon = \mathrm{e}^{-\frac{i}{\hbar}\epsilon \mathrm{Op}^{\mathrm{W}} g} M(A) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar.$$

This is the quantum map we wish to study. It is naturally related to the discrete Hamiltonian dynamics on $\mathbb{T}^2$ obtained by iterating $\Phi_\epsilon = \phi_\epsilon \circ A$. It acts on the N dimensional spaces $\mathcal{H}_\hbar$ and we are interested in the behaviour of its eigenfunctions and eigenvalues in the $N \to \infty$ limit:

$$U_\epsilon \psi_j^{(N)} = \mathrm{e}^{i\theta_j^{(N)}} \psi_j^{(N)}, \quad j = 1 \ldots N.$$

WHAT IS KNOWN?

THEOREM 1 (Bouzouina-DB 96) Let $\epsilon \geq 0$ and small.

Then, for "almost all" sequences $\psi_N \in \mathcal{H}_\hbar$, so that $U_\epsilon \psi_N = e^{i\theta_N}\psi_N$,

$$\langle \psi_N, \mathrm{Op}^{\mathrm{W}} f \psi_N \rangle \overset{N \to +\infty}{\to} \int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2)$$

THEOREM 2 (Faure-Nonnenmacher-DB 03) Let $\epsilon = 0$. Let $0 \leq \alpha \leq \frac{1}{2}$, then there exists $N_k \to \infty$ and eigenfunctions $\psi_{N_k} \in \mathcal{H}_{N_k}$ so that

$$\langle \psi_{N_k}, \mathrm{Op}^{\mathrm{W}} f \psi_{N_k} \rangle \overset{N_k \to +\infty}{\to} \alpha f(0) + (1-\alpha)\int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2).$$

THEOREM 3 (Bonechi-DB 03, Faure-Nonnenmacher 04) Let $\epsilon = 0$. Let $\alpha > \frac{1}{2}$. Then there does not exist $N_k \to \infty$ and eigenfunctions $\psi_{N_k} \in \mathcal{H}_{N_k}$ so that

$$\langle \psi_{N_k}, \mathrm{Op}^{\mathrm{W}} f \psi_{N_k} \rangle \overset{N_k \to +\infty}{\to} \alpha f(0) + (1-\alpha)\int_{\mathbb{T}^2} f(x)dx, \quad \forall f \in C^\infty(\mathbb{T}^2).$$

THEOREM 4,5,6, ... $\epsilon = 0$ Degli Esposti-Graffi-Isola (95), Kurlberg-Rudnick (00-01), ...

A QUESTION : Which of Theorems 2,3,4,... has a chance to generalize to $\epsilon \neq 0$?

AN ANSWER?

THEOREM 1, the “Schnirelman theorem” is robust. Its proof is an adaptation to the present context of known arguments: Schnirelman (74), Zelditch (87,95), Colin de Verdière (85), Helffer-Martinez-Robert (87). It always works if the classical system is ergodic. That's why here $\epsilon \neq 0$ is allowed. It holds for higher dimensional tori as well.

THEOREM 2 uses one very special arithmetic property of the problem, and I don't think it should survive perturbation ($\epsilon \neq 0$).

THEOREM 4,5,6, . . . rely on the arithmetic properties of the system . . .

THEOREM 3 ought to hold for $\epsilon \neq 0$ as well. Why?

Theorem 3 : the main ingredient

THEOREM (Bonechi-DB 03)"Congregate, and thou shalt be spread"

Let $\epsilon = 0$. Let $a_0 \in \mathbb{T}^2$. If $\varphi_N \in \mathcal{H}_\hbar$ is some sequence (not necessarily eigenfunctions!) with the property that, for all $f \in C^\infty(\mathbb{T}^2)$

$$\langle \varphi_N, \mathrm{Op}^{\mathrm{W}} f \varphi_N \rangle \overset{N \to +\infty}{\to} f(a_0)$$

then there exists a sequence of times $t_N \to \infty$ so that

$$\langle \varphi_N, U_0^{-t_N} \mathrm{Op}^{\mathrm{W}} f \, U_0^{t_N} \varphi_N \rangle \overset{N \to +\infty}{\to} \int_{\mathbb{T}^2} f(x) dx.$$

Remark: The statement involves the simultaneous limit

"$\hbar$ to zero, time to infinity".

The time scales t_N involved are logarithmic in $\hbar$. So if you want to prove something like this for $\epsilon \neq 0$ (perturbed automorphisms), you need to be able to control $U_\epsilon^{t_N}$ at such time scales. For the Schnirelman theorem, that's not needed since there, you take $\hbar \to 0$ first, and then $t \to \infty$.

THE EGOROV THEOREM 1

Recall: We are given $A \in \mathrm{SL}(2,\mathbb{Z})$, $g \in C^\infty(\mathbb{T}^2)$ and $\epsilon \geq 0$

The Lyapounov exponent γ_0 is defined by $Av_\pm = \mathrm{e}^{\pm\gamma_0} v_\pm$.

The classical dynamics is $\Phi_\epsilon = \phi_\epsilon \circ A$.

The quantum dynamics is $U_\epsilon = \mathrm{e}^{-\frac{i\epsilon}{\hbar}\mathrm{Op}^{\mathrm{W}} g} M(A)$.

"THEOREM" Let $f \in C^\infty(\mathbb{T}^2)$. There exists

$$\Gamma_\epsilon \geq \gamma_0, \quad \lim_{\epsilon\to 0}\Gamma_\epsilon = \gamma_0,$$

such that, for all $0 < \nu$, for all $t \in \mathbb{Z}$ such that

$$0 \leq |t| \leq (\frac{2}{3\Gamma_\epsilon} - \nu)|\ln\hbar|$$

$$U_\epsilon^{-t}\,\mathrm{Op}^{\mathrm{W}} f\, U_\epsilon^t - \mathrm{Op}^{\mathrm{W}}(f \circ \Phi_\epsilon^t) \stackrel{\hbar\to 0}{\longrightarrow} 0.$$

THE EGOROV THEOREM 2

Recall: Given $A \in \mathrm{SL}(2,\mathbb{Z})$, $g \in C^\infty(\mathbb{T}^2)$ and $\epsilon \geq 0$. $Av_\pm = \mathrm{e}^{\pm\gamma_0} v_\pm$

Dynamics : $\Phi_\epsilon = \phi_\epsilon \circ A$ and $U_\epsilon = \mathrm{e}^{-\frac{i\epsilon}{\hbar}\mathrm{Op}^{\mathrm{W}} g} M(A)$.

THEOREM Let $f, g \in C^\infty(\mathbb{T}^2)$ be bounded and analytic in a strip of width δ_0. There exists

$$\Gamma_\epsilon \geq \gamma_0, \quad \lim_{\epsilon\to 0} \Gamma_\epsilon = \gamma_0,$$

such that, for all $0 < \nu$, there exists $J_0 > 0$ so that for all $J \geq J_0$ and for all $t \in \mathbb{Z}$,

$$U_\epsilon^{-t}\,\mathrm{Op}^{\mathrm{W}} f\, U_\epsilon^t - \mathrm{Op}^{\mathrm{W}}(f \circ \Phi_\epsilon^t) = \hbar \sum_{1 \leq j < J} \hbar^{j-1} \mathrm{Op}^{\mathrm{W}}(\mathcal{L}_j^t f) + \hbar^J \rho_J(f, \epsilon, \hbar, t)$$

where,

$$\| \partial^\beta \mathcal{L}_j^t f \|_\infty \leq \| f \|_{\infty,\delta_0} C_{j,\beta} \mathrm{e}^{t\Gamma_\epsilon(|\beta| + \frac{3}{2}j)}, \quad \forall t \in \mathbb{Z}.$$

and, for $0 \leq |t| \leq (\frac{2}{3\Gamma_\epsilon} - \nu)|\ln \hbar|$

$$\| \hbar^J \rho_J(f, \epsilon, \hbar, t) \|_{\mathcal{H}_\hbar \to \mathcal{H}_\hbar} \to 0.$$

COHERENT STATE PROPAGATION

Constructing coherent states: the recipe

Ingredients : $\varphi \in \mathcal{S}(\mathbb{R}), 0 < \sigma < 1, a \in \mathbb{R}^2.$

Construction: For $y \in \mathbb{R}$,

$$\varphi_\hbar(y) = \frac{1}{\hbar^{\sigma/2}}\varphi(\frac{y}{\hbar^\sigma}), \quad \varphi_{a,\hbar}(y) = U(a)\varphi_\hbar(y) \in \mathcal{S}(\mathbb{R}).$$

With P defined as $P = \sum_{n\in\mathbb{Z}^2}(-1)^{n_1 n_2}U(n)$, it is a fact that

$$P\mathcal{S}(\mathbb{R}) = \mathcal{H}_\hbar.$$

Define finally

$$\tilde{\varphi}_{a,\hbar} = S\varphi_{a,\hbar} \in \mathcal{H}_\hbar.$$

Done.

PROPOSITION :

$$\langle\tilde{\varphi}_{a,\hbar}, \mathrm{Op}^{\mathrm{W}} f\ \tilde{\varphi}_{a,\hbar}\rangle_{\mathcal{H}_\hbar} \stackrel{N\to+\infty}{\to} f(a)$$

Evolving coherent states:

THEOREM

$$\langle \tilde{\varphi}_{a,\hbar}, U_\epsilon^{-t}\, \mathrm{Op}^{\mathrm{W}} f\, U_\epsilon^{t}\, \tilde{\varphi}_{a,\hbar}\rangle_{\mathcal{H}_\hbar} \stackrel{N\to+\infty}{\to} \int_{\mathbb{T}^2} f(x)dx.$$

provided

$$(\min(\sigma, 1-\sigma)+\nu)\,\frac{1}{\gamma_\epsilon}|\ln\hbar| \le |t| \le (\frac{2}{3\Gamma_\epsilon}-\nu)|\ln\hbar|.$$

Recall that

$$\gamma_\epsilon \le \gamma_0 \le \Gamma_\epsilon$$

and that

$$\lim_{\epsilon\to 0}\gamma_\epsilon = \gamma_0 = \lim_{\epsilon\to 0}\Gamma_\epsilon$$

so that the time window is non-trivial.

BACK TO THEOREM 3

DEFINITION A sequence $\varphi_N \in \mathcal{H}_\hbar$ is said to localize at $a \in \mathbb{T}^2$ if

$$\langle \varphi_n, \mathrm{Op}^{\mathrm{W}} f\, \varphi_N \rangle \overset{N \to +\infty}{\longrightarrow} f(a), \qquad \forall f \in C^\infty(\mathbb{T}^2).$$

This implies there exists $r_\hbar \to 0$ so that

$$\int_{|x-a| \geq r_\hbar} |\langle \varphi_N, \tilde{\varphi}_x \rangle|^2 \frac{dx}{2\pi\hbar} \overset{N \to +\infty}{\longrightarrow} 0.$$

THEOREM Let U_ϵ be as before. There exists $\delta_0 > 0$ with the following property. Let ψ_N be a sequence of eigenfunctions of U_ϵ that localizes at some point $a \in \mathbb{T}^2$. Then $r_\hbar \geq \hbar^{\frac{1}{2} - \delta_0}$.

This gives some weak control on the way eigenfunctions may concentrate. If they do so, they must do it slowly! We expect to be able to improve the exponent $\frac{1}{2} - \delta_0$ (See also the talk by Nonnenmacher at this conference).