On the number of lattice points in a thin annulus

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joint with:

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Count the number of integer lattice points in a circle of radius t.

$$N(t) = \#\left\{(x,y) \in \mathbb{Z}^2 \ : \ x^2 + y^2 \leq t^2\right\}$$

Gauss:

$$N(t) = \pi t^2 + O(t)$$

The error term has been studied by many people, starting with Sierpinski. The current world record (Huxley, 2003) is $O(t^{131/208})$.

Hardy conjectured: "It is not unlikely that" the error term is $O(t^{1/2+\varepsilon})$

Heath-Brown showed that

$$\frac{N(t) - \pi t^2}{\sqrt{t}}$$

has a probability distribution. This distribution is not Gaussian.

Study the number of lattice points in an annulus:

$$N(t, \rho) = N(t + \rho) - N(t)$$

where we allow $\rho = \rho(t)$. Set

$$S(t, \rho) = \frac{N(t + \rho) - N(t) - \pi(2\rho t + \rho^2)}{\sqrt{t}}$$

Various regimes:

- If $\rho \to \infty$, then $S(t, \rho)$ has a non-Gaussian distribution (Bleher, Lebowitz)
- If $\rho \to 0$ but $t\rho \to \infty$, then Bleher and Lebowitz conjectured $S(t, \rho)$ has a Gaussian distribution. This will be proved in this talk if $\rho \to 0$ sufficiently slowly.
- If $\rho \to 0$ such that the area of the annulus, $t\rho$, is a constant, then $S(t, \rho)$ might have a discrete distribution. For sufficiently generic ellipses this is expected to be the Poisson distribution.

Theorem 1 (Hughes and Rudnick)
If $\rho \to 0$ but $\rho \gg T^{-\varepsilon}$ for all $\varepsilon > 0$, then for any interval $\mathcal{A}$

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ t \in [T, 2T] \ : \ \frac{S(t,\rho)}{\sigma} \in \mathcal{A} \right\} = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{A}} e^{-x^2/2} \, \mathrm{d}x$$

where $\sigma^2 = 16\rho \log \frac{1}{\rho}$.

This theorem has been generalized by Igor Wigman to elliptic annuli, whose aspect ratio is transcendental and strongly diophantine.

- Replace the sharp counting function $N(t)$ by a smooth counting function $\widetilde{N}(t)$ and smooth normalized remainder term $\widetilde{S}(t,\rho)$.
- Compute the moments of $\widetilde{S}(t,\rho)$ when t is averaged with respect to a smooth measure.
- Its variance is $\sigma^2 = 16\rho\log(1/\rho)$ if $\rho \to 0$ but $\rho \gg T^{-1+\varepsilon}$.
- The m^{th} moment of $\widetilde{S}(t,\rho)/\sigma$ converges to that of a standard normal random variable.
- Thus $\widetilde{S}(t,\rho)$ has a normal distribution if $\rho \to 0$ but $\rho \gg T^{-\varepsilon}$ for all $\varepsilon > 0$.
- The variance of the difference $(S(t,\rho) - \widetilde{S}(t,\rho))/\sigma$ goes to zero.
- Hence $S(t,\rho)/\sigma$ has a normal distribution with respect to the smooth measure.
- Use an approximation argument to pass from smooth measures to the Lebesgue measure.

Smoothing the edges of the circle

Let $\widehat{\psi}$ be a smooth function with compact support, and $\widehat{\phi}(0) = 1$.

Define a rotationally symmetric function Ψ on $\mathbb{R}^2$ by setting $\widehat{\Psi}(\vec{y}) = \widehat{\psi}(|\vec{y}|)$.

For $\delta > 0$ set

$$\Psi_\delta(\vec{x}) = \frac{1}{\delta^2}\Psi(\frac{\vec{x}}{\delta})$$

Let χ be the indicator function of the unit disc.

Now set $\chi_\delta = \chi * \Psi_\delta$ to be the convolution of χ and Ψ_δ.

Define a smooth counting function by

$$\widetilde{N}(t) = \sum_{\vec{n}\in\mathbb{Z}^2} \chi_\delta(\frac{\vec{n}}{t})$$

This counts lattice points in a "fuzzy circle" of radius about t, with fuzziness about $t\delta =: 1/\sqrt{M}$.

Poisson summation.

As $t \to \infty$,

$$\widetilde{N}(t) = \pi t^2 - \frac{\sqrt{t}}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \cos(2\pi t\sqrt{n} + \tfrac{1}{4}\pi)\widehat{\psi}(\sqrt{\frac{n}{M}}) + O\left(\frac{1}{\sqrt{t}}\right)$$

where $r(n)$ is the number of ways of writing $n = x^2 + y^2$ as the sum of two squares.

$$\widetilde{S}(t,\rho) = \frac{\widetilde{N}(t+\rho) - \widetilde{N}(t) - \pi(2t\rho + \rho^2)}{\sqrt{t}}$$

then for $t > 1$ and $\rho < 1$,

$$\widetilde{S}(t,\rho) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{r(n)}{n^{3/4}} \sin\left(\pi\rho\sqrt{n}\right) \times \sin\left(2\pi(t + \tfrac{1}{2}\rho)\sqrt{n} + \tfrac{\pi}{4}\right) \widehat{\psi}(\sqrt{\frac{n}{M}}) + O\left(\frac{1}{t}\right)$$

Note that we have three independent variables.

The variable t, which we always consider to be large, is the radius of the annulus. We average t over $[T, 2T]$.

The width of the annulus is ρ. Since we want a thin annulus, we let $\rho \to 0$, and Gaussian behaviour is not seen if this condition does not hold.

The annulus does not have sharp sides, but smoothed edges, and the third independent variable is M; the larger M is, the sharper the annulus' sides are (in the sense that it better approximates the indicator function). We must have $\rho\sqrt{M} \to \infty$ in order for the annulus to have some width, and not be "just sides". That is, the annulus shouldn't be too smooth.

Lemma 1 *If $M = O(T^{2(1-\varepsilon)})$ for fixed $\varepsilon > 0$, then the variance of $\widetilde{S}(t,\rho)$ is asymptotic to*

$$\sigma^2 := \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}} \sin^2\left(\pi\rho\sqrt{n}\right) \widehat{\psi}^2\left(\sqrt{\frac{n}{M}}\right)$$

If $\rho \to 0$ but $\rho\sqrt{M} \to \infty$, then

$$\sigma^2 \sim 16\rho\log(1/\rho)$$

Proof. Expand out $\widetilde{S}(t,\rho)^2$ and average t between T and $2T$. If $M = O(T^{2(1-\varepsilon)})$ then we can justify the restriction of the sum to diagonal terms, which gives, for any $B > 0$,

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}} \sin^2\left(\pi\rho\sqrt{n}\right) \widehat{\psi}^2\left(\sqrt{\frac{n}{M}}\right) + O\left(T^{-B}\right)$$

Define σ^2 to be the infinite sum above.

To find the asymptotics as $\rho \to 0$, we use a formula of Ramanujan:

$$\sum_{n \le X} r(n)^2 = 4X \log X + O(X) \, .$$

Which leads to

$$\sigma^2 \sim \rho \log(1/\rho) \frac{32}{\pi^2} \int_0^\infty \frac{\sin^2(\pi y)}{y^2} \widehat{\psi}^2 \left(\frac{y}{\rho\sqrt{M}} \right) \, \mathrm{d}y$$

If $\rho\sqrt{M} \to \infty$ then $\widehat{\psi}(y/\rho\sqrt{M}) \sim 1$, and so

$$\sigma^2 \sim 16\rho \log(1/\rho)$$

Higher moments:

Expanding $\widetilde{S}(t,\rho)^m$ and averaging over t, we can show only the diagonal terms contribution if $M = O\left(T^{2(1-\varepsilon)/(2^{m-1}-1)}\right)$ for fixed $\varepsilon > 0$.

Furthermore, if $\rho \to 0$ such that $\rho\sqrt{M} \to \infty$, then for arbitrary $\varepsilon' > 0$,

$$\frac{\langle \widetilde{S}(t,\rho)^m \rangle}{\sigma^m} = \begin{cases} \frac{m!}{2^{m/2}(m/2)!} + O\left(\rho^{-1+\varepsilon'}\right) & \text{if } m \text{ is even} \\ O\left(\rho^{-1+\varepsilon'}\right) & \text{if } m \text{ is odd} \end{cases}$$

Essential to the proof is a theorem of Besicovitch: If $q_1, \ldots, q_m$ are distinct square-free integers, then $\sqrt{q_1}, \ldots, \sqrt{q_m}$ are linearly independent over the rationals.

Using a truncated Poisson summation formula for the unsmoothed $S(t,\rho)$, we can show that if $\rho \to 0$ but $\rho\sqrt{M} \to \infty$ as $T \to \infty$, and $M = O(T^{2(1-\varepsilon)})$, then

$$\langle |S(t,\rho) - \widetilde{S}(t,\rho)|^2 \rangle \ll \frac{\log M}{\sqrt{M}}$$

as $T \to \infty$.

Note that this is much smaller than $1/\sigma^2$.

Tchebychev's inequality therefore gives that for fixed $\eta > 0$,

$$\mathbb{P}\left\{ \left| \frac{S(t,\rho)}{\sigma} - \frac{\widetilde{S}(t,\rho)}{\sigma} \right| > \eta \right\} \to 0$$

Since $\widetilde{S}(t,\rho)/\sigma$ weakly converges to a standard normal distribution as $T \to \infty$ when t is averaged around T, this lemma implies that $S(t,\rho)/\sigma$ must also weakly converge to a standard normal distribution.

That is, if $\rho \to 0$ but $\rho = O(T^{-\varepsilon})$ for all $\varepsilon > 0$ as $T \to \infty$, then for any interval $\mathcal{A}$,

$$\mathbb{P}\left\{\frac{S(t,\rho)}{\sigma} \in \mathcal{A}\right\} \to \frac{1}{\sqrt{2\pi}}\int_{\mathcal{A}} e^{-x^2/2}\,\mathrm{d}x \qquad (1)$$

where $\sigma^2 = 16\rho\log(1/\rho)$.