Semiclassical evidence for universal spectral correlations in quantum chaos

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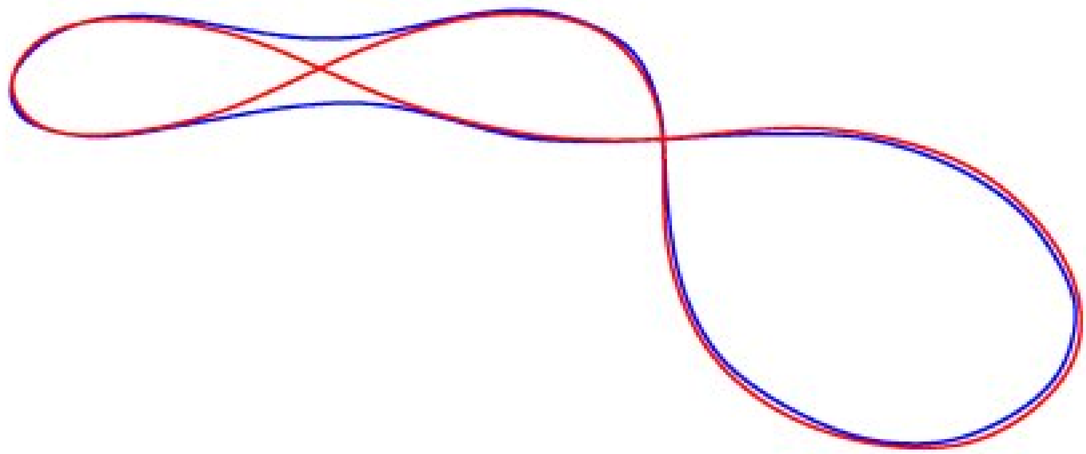


Universality in spectral statistics of quantum systems with a chaotic classical limit $\Longleftrightarrow$ properties of periodic orbits

Universal spectral statistics

Quantum spectrum

$$\hat{H}\,\Psi_n = E_n\,\Psi_n$$

Conjectures on statistical distributions of energy levels E_n, $n = 0, 1, \ldots,$ depending on the classical limit of the system

- Integrable systems $\Longrightarrow$ Poisson process (Berry, Tabor (1977))
- Chaotic systems $\Longrightarrow$ RMT (Bohigas, Giannoni, Schmit (1984))

Example: Riemann surfaces with constant negative curvature ($K = 1$).

Properties: Uniformly hyperbolic, no focussing of trajectories, Selberg trace formula

Quantum problem

$$-\Delta\,\Psi_n = E_n\,\Psi_n$$

Momenta: $E_n = p_n^2 + 1/4$

For "generic" Riemann surfaces one expects statistical distributions of energy levels to agree with those of the Gaussian Orthogonal Ensemble (GOE) of random matrices.

We consider systems in which multiplicity of lengths of periodic orbits is typically 2.

The number variance

Consider

$$n(E,S) = \#\left\{E_n | E - \frac{S}{2\bar{d}(E)} < E_n < E + \frac{S}{2\bar{d}(E)}\right\}$$

where $\bar{d}(E) \sim \frac{A}{4\pi}$, $E \to \infty$, is the mean density of eigenstates.

Let $\langle \ldots \rangle_E$ denote a local average over a window ΔE around E. By construction

$$\langle n(E,S) \rangle_E \approx S$$

The number variance is defined as

$$\Sigma(E,S) = \langle [n(E,S) - S]^2 \rangle_E$$

Expectation

$$\lim_{E \to \infty} \Sigma(E,S) = \Sigma^{\mathsf{GOE}}(S)$$

where $1 \ll \Delta E \ll E$

Spectral form factor

The spectral form factor is defined as

$$\tilde{K}(E,\tau) = 1 + \frac{1}{2}\int_{-\infty}^{\infty} dS \, \frac{d^2}{dS^2}\Sigma(E,S) \, \exp(i2\pi\tau S)$$

Applying Selberg trace formula yields

$$K(p,\tau) = \left\langle \frac{1}{pA}\sum_{\gamma,\gamma'} A_\gamma A_{\gamma'} \, e^{ip(L_\gamma - L_{\gamma'})} \, \delta\left(L - \frac{L_\gamma + L_{\gamma'}}{2}\right)\right\rangle_{p,L}$$

where the sums run over all periodic orbits

$$\tau = \frac{L}{pA}\,, \qquad A_\gamma = \frac{L_\gamma}{2r_\gamma \sinh L_\gamma/2}$$

Semiclassical limit ($p \to \infty$, $\tau = const.$) implies $L \to \infty$.

Expect

$$\lim_{p\to\infty} K(p,\tau) = K^{\mathsf{GOE}}(\tau)$$

Choice of averaging

Gaussian averaging in p and L

$$\langle f(x) \rangle_x = \int_{-\infty}^{\infty} dx' \, \frac{f(x')}{\sqrt{2\pi\Delta x^2}} \exp\left(\frac{(x-x')^2}{2\Delta x^2} \right)$$

Choosing $\Delta p \, \Delta L = 1/2$ leads to

$$K(p,\tau) = \frac{\Delta p \sqrt{2}}{pA} \left| \sum_\gamma A_\gamma \exp\{ipL_\gamma - \Delta p^2 \, (L - L_\gamma)^2\} \right|^2$$

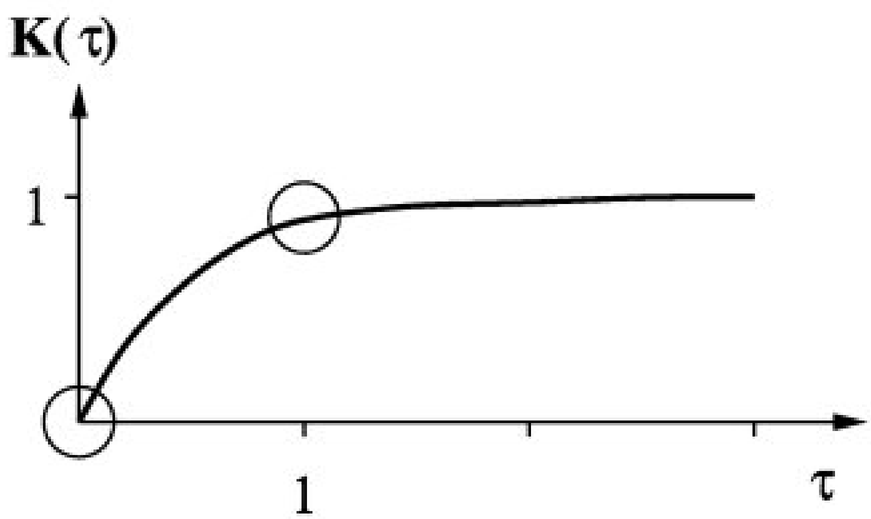
This is the form factor of the momentum-spectrum

Semiclassical limit: $p \to \infty$ with $p^{-1} \ll \Delta p \ll p^{1/2}$.

The GOE form factor

The form factor of the Gaussian Orthogonal Ensemble

$$K^{\mathsf{GOE}}(\tau) = \begin{cases} 2\tau - \tau \log(1+2\tau) & \text{if } \tau < 1 \\ 2 - \tau \log \frac{2\tau+1}{2\tau-1} & \text{if } \tau > 1 \end{cases}$$



Expansion for small values of τ

$$K^{\mathsf{GOE}}(\tau) = 2\tau - 2\tau^2 + 2\tau^3 + \dots$$

The diagonal approximation

Diagonal approximation (Berry (1985))

$$K^{(1)}(\tau) = \left\langle \frac{2}{pA} \sum_{\gamma} A_{\gamma}^{2}\, \delta(L - L_{\gamma}) \right\rangle$$

Amplitudes

$$A_{\gamma}^{2} \sim \frac{L_{\gamma}^{2}}{e^{L_{\gamma}}} \quad \text{as} \quad L_{\gamma} \to \infty$$

Prime geodesic theorem

$$N(L) = \#\{L_{\gamma} \leq L\} \sim \frac{e^{L}}{L} \quad \text{as} \quad L \to \infty$$

$$\bar{\rho}(L) \sim \frac{d}{dL}\left(\frac{e^{L}}{L}\right) \sim \frac{e^{L}}{L}$$

$$\sum_{\gamma} f(L_{\gamma}) \to \int dL'\, \bar{\rho}(L')\, f(L')$$

Result (Hannay, Ozorio de Almeida (1984), Berry (1985))

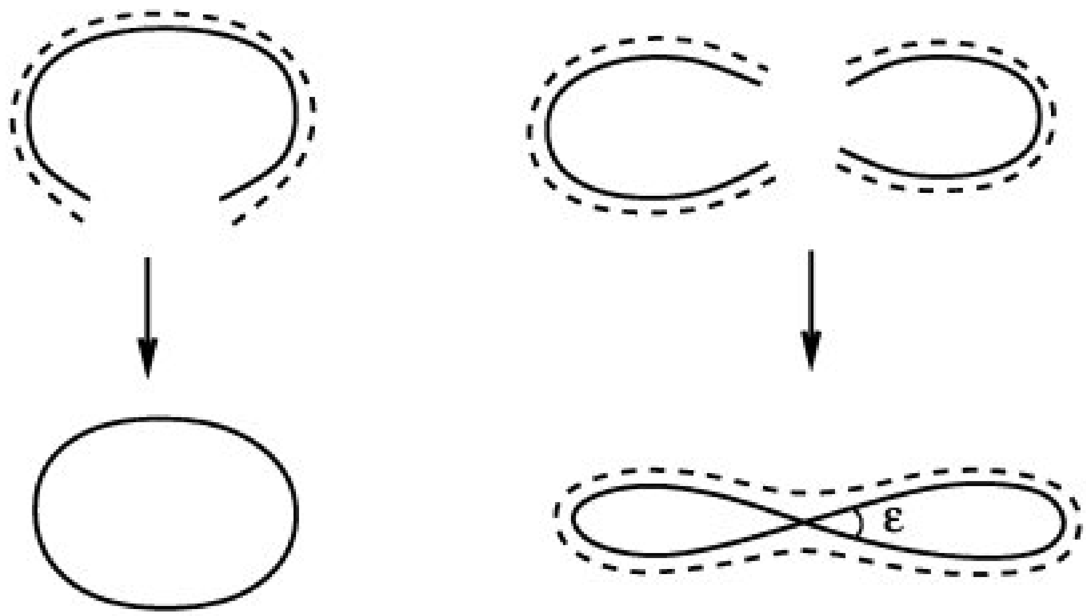
$$K^{(1)}(\tau) \sim \left\langle \frac{2}{pA} \int^{\infty} dL'\, L'\, \delta(L - L') \right\rangle$$

$$K^{(1)}(\tau) \sim 2\tau$$

Off-diagonal contributions

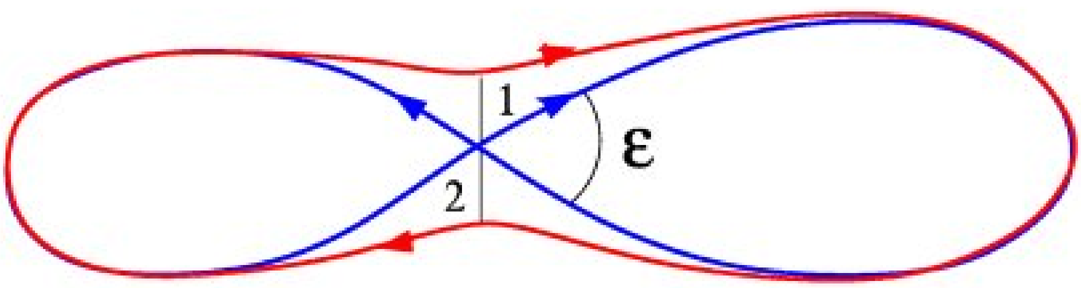
$$K(\tau) = \left\langle \frac{1}{pA} \sum_{\gamma,\gamma'} A_\gamma A_{\gamma'} \, e^{ip(L_\gamma - L_{\gamma'})} \, \delta\left(L - \frac{L_\gamma + L_{\gamma'}}{2}\right) \right\rangle_{p,L}$$

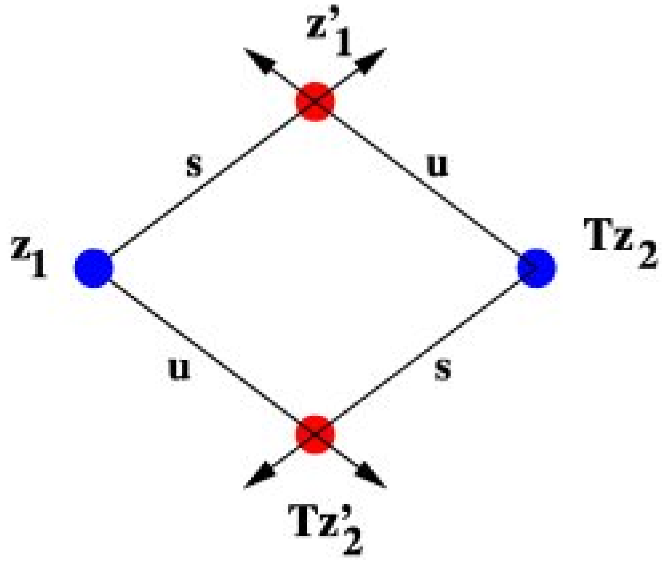
Orbits have to be correlated (Argaman et al. (1993))



(M.S., Richter (2001), M.S. (2002))

Two-loop orbits





Length difference

$$\Delta L \sim \frac{\epsilon^2}{2}$$

Symbolic dynamics (Braun et al. (2002))

$$\Delta L \sim -4 \log \cos \frac{\epsilon}{2}$$

Number of self-intersections

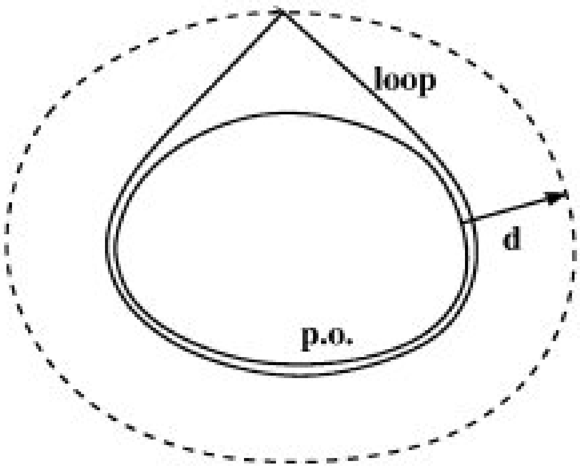
The average number of self-intersections along a trajectory of length L with angle in $[\epsilon, \epsilon + d\epsilon]$: $P(\epsilon, L)\, d\epsilon$

$$P(\epsilon, L) = \int_0^L dl\ (L - l)\ \sin\epsilon\, p(\epsilon, l)$$

Probability to form a loop

$$p(\epsilon, l) = \frac{1}{2\pi A} \int d^2 q_0\, d\theta_0\ \delta(\mathbf{q}(l) - \mathbf{q}_0)\ \delta(\theta(l) - \theta_0 + \pi - \epsilon)$$

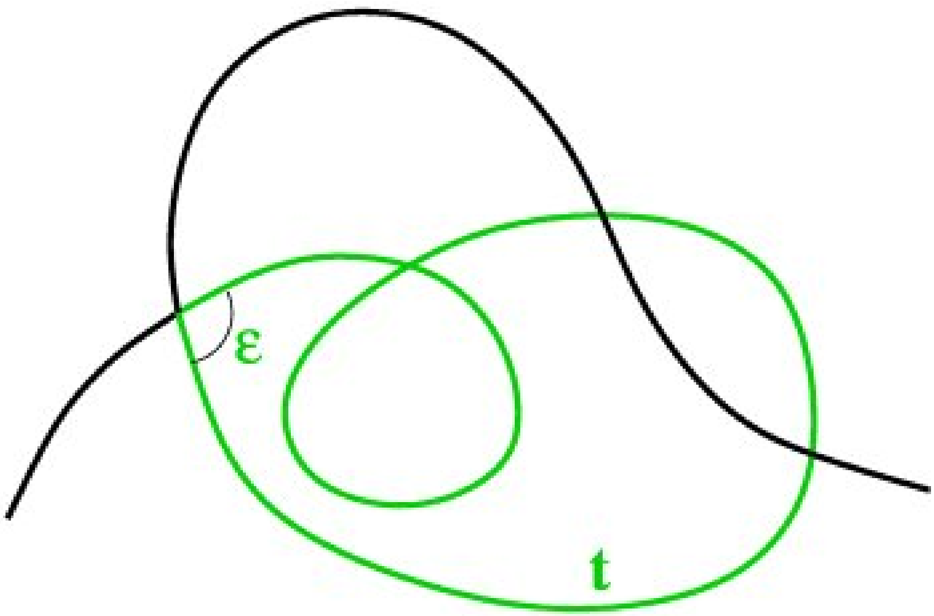
can be expressed in terms of loops

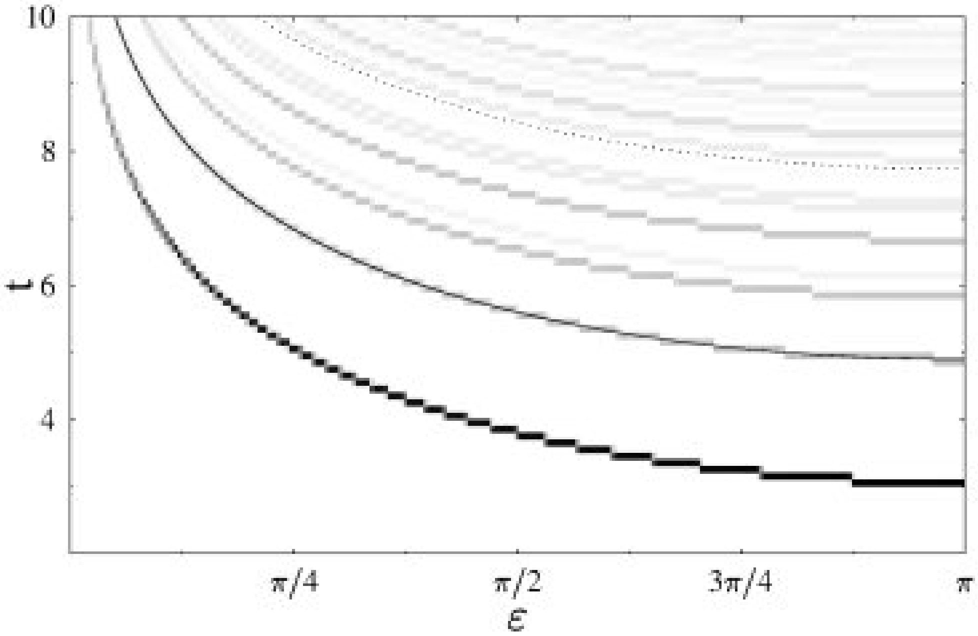


Relation to periodic orbits

$$\cosh\frac{l_\gamma}{2}\ \sin\frac{|\epsilon|}{2} = \cosh\frac{L_\gamma}{2}$$

Numerical result for loops





The logarithmic divergence

$$\cosh \frac{l_\gamma}{2}\ \sin \frac{|\epsilon|}{2} = \cosh \frac{L_\gamma}{2}$$

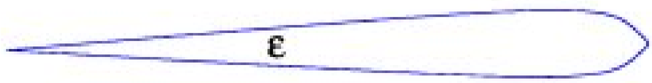
For long periodic orbits

$$l_\gamma(\epsilon) \sim L_\gamma - 2 \log \sin \frac{|\epsilon|}{2} \quad \text{as} \quad L_\gamma \to \infty$$

and small angles

$$l_\gamma(\epsilon) \sim L_\gamma - 2 \log \frac{|\epsilon|}{2} \quad \text{as} \quad L_\gamma \to \infty$$

Consider a loop with small angle ϵ



Minimal loop length estimate:

$$|\epsilon| \exp(l_{\text{min}}/2) = \mathcal{O}(1)$$

leads to

$$l_{\text{min}} \sim -2 \log c\epsilon$$

Asymptotic approximation for $P(\epsilon, L)$

Periodic orbit expansion of $p(\epsilon, l)$

$$p(\epsilon, l) = \frac{1}{2\pi A} \sum_{\gamma} B_\gamma(\epsilon)\ \delta(l - l_\gamma(\epsilon))$$

where

$$B_\gamma(\epsilon) = \frac{L_\gamma}{4r_\gamma \sqrt{\sinh^2 L_\gamma/2\,(\sinh^2 L_\gamma/2 + \cos^2 \epsilon/2)}}$$

Hence

$$P(\epsilon, L) = \frac{\sin\epsilon}{2\pi A} \sum_{l_\gamma(\epsilon) < L} B_\gamma(\epsilon)\,(L - l_\gamma(\epsilon))$$

where $l_\gamma(\epsilon) \sim L_\gamma - 2\log\sin\frac{|\epsilon|}{2}$ as $L_\gamma \to \infty$

Using prime geodesic theorem

$$P(\epsilon, L) \sim \frac{\sin\epsilon}{2\pi A} \int^{L + 2\log\sin|\epsilon|/2} dL' \left(L - L' + 2\log\sin\frac{|\epsilon|}{2}\right)$$

$$\sim \frac{L^2}{2\pi A} \sin\epsilon \qquad \text{as} \qquad L \to \infty$$

Contribution to the formfactor

$$K(\tau) = \left\langle \frac{1}{pA} \sum_{\gamma,\gamma'} A_\gamma A_{\gamma'} \, e^{ip(L_\gamma - L_{\gamma'})} \, \delta\left(L - \frac{L_\gamma + L_{\gamma'}}{2}\right) \right\rangle$$

Contribution from two-loop orbits

$$K^{(2)}(\tau) \sim \left\langle \frac{4}{pA} \mathrm{Re} \int d\epsilon \sum_\gamma A_\gamma^2 e^{ip\Delta L(\epsilon)} \, \delta(L - L_\gamma) \, P_\gamma(\epsilon) \right\rangle$$

Uniform distribution of periodic orbits in phase space

$$`\sum_\gamma A_\gamma^2 \, \delta(L - L_\gamma) \, P_\gamma(\epsilon) \sim L \, P(L, \epsilon)' \qquad \text{as} \qquad L \to \infty$$

Hence

$$\begin{aligned} K^{(2)}(\tau) &\sim \left\langle \frac{4L}{pA} \mathrm{Re} \int d\epsilon \, e^{ip\Delta L(\epsilon)} \, P(L, \epsilon) \right\rangle \\ &\sim \frac{2L^3}{\pi p A^2} \mathrm{Re} \int_0^\infty e^{ip\epsilon^2/2} \, \epsilon \\ &\sim \frac{2\tau^3}{\pi} pA \, \mathrm{Re} \, i \end{aligned}$$

Semiclassical limit $p \to \infty$ implies $L \propto p \to \infty$, $\epsilon \propto \frac{1}{\sqrt{p}} \to \infty$

Asymptotic expansion of $P(\epsilon, L)$

$$P(\epsilon, L) = \frac{\sin\epsilon}{2\pi A} \sum_{l_\gamma(\epsilon)<L} B_\gamma(\epsilon)\,(L - l_\gamma(\epsilon))$$

$$\sim \frac{\sin\epsilon}{2\pi A} \sum_{L_\gamma<L_*} B_\gamma(\epsilon)\,(L - l_\gamma(\epsilon))$$

$$+ \frac{1}{2\pi A} \int_{L_*}^{L+2\log\sin|\epsilon|/2} dL' \, \left(L - L' + 2\log\sin\frac{|\epsilon|}{2}\right)$$

Leads to

$$P(\epsilon, L) \sim \frac{L^2}{4\pi A} \sin|\epsilon| \left(1 + \frac{2C(\epsilon)}{L} + \frac{4}{L}\log\sin\frac{|\epsilon|}{2}\right)$$

as $L \to \infty$, where

$$C(\epsilon) = \lim_{L_*\to\infty} \left(\sum_{L_\gamma<L_*} B_\gamma(\epsilon) - L_* \right)$$

In the limit $L \to \infty$ and $\epsilon \propto L^{-1/2} \to 0$

$$P(\epsilon, L) \sim \frac{L^2}{4\pi A} \sin|\epsilon| \left(1 + \frac{4}{L}\log c|\epsilon|\right)$$

where $c = \frac{1}{2}\exp(-C(0)/2)$

The τ^2-term

$$K^{(2)}(\tau) \sim \left\langle \frac{4}{pA} \mathrm{Re} \int d\epsilon \sum_\gamma A_\gamma^2 e^{ip\Delta L(\epsilon)} \, \delta(L - L_\gamma) \, P_\gamma(\epsilon) \right\rangle$$

Uniform distribution of periodic orbits in phase space

$$\text{‘} \sum_\gamma A_\gamma^2 \, \delta(L - L_\gamma) \, P_\gamma(\epsilon) \sim \frac{L^3}{4\pi A} \sin|\epsilon| \, \left(1 + \frac{4}{L} \log c|\epsilon|\right) \text{’}$$

as $L \to \infty$ and $\epsilon \propto L^{-1/2} \to 0$

$$K^{(2)}(\tau) \sim \frac{8L^2}{\pi p A^2} \mathrm{Re} \int_0^\infty d\epsilon \, e^{ip\epsilon^2/2} \, \epsilon \, \log(c\epsilon)$$

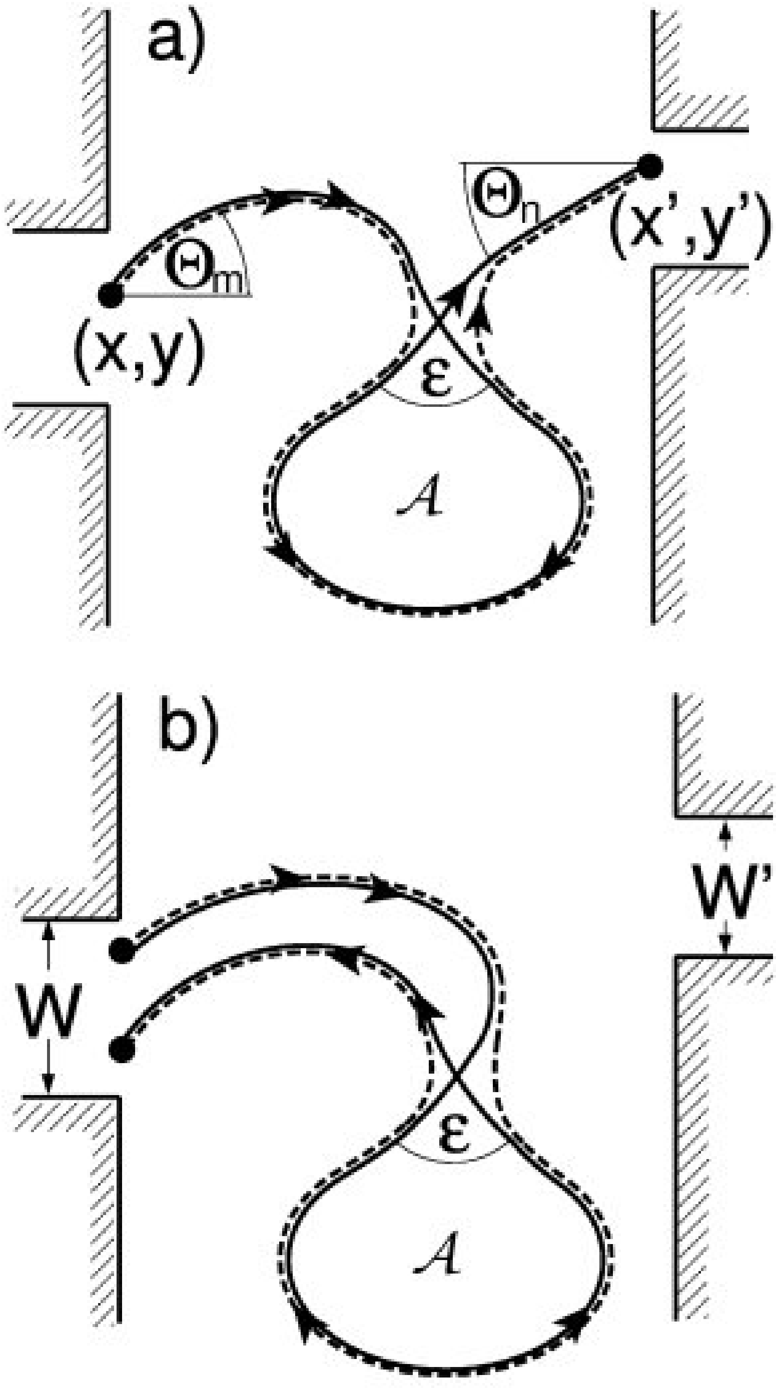
$$\epsilon = e^{i\pi/4} \, \epsilon'$$

$$K^{(2)}(\tau) \sim -\frac{8L^2}{\pi p A^2} \mathrm{Im} \int_0^\infty d\epsilon' \, e^{-p\epsilon'^2/2} \, \epsilon' \, \log(c\epsilon' e^{i\pi/4})$$

Result

$$K^{(2)}(\tau) \sim -2\tau^2$$

Transport problem



Richter, M.S. (2002)

Further work

Quantum graphs:

Berkolaiko, Schanz, Whitney (2002,2003); Berkolaiko (2004); Gnutzmann, Altland (2004)

Nonuniformly hyperbolic systems:

Turek, Richter (2003); Spehner (2003); Müller (2003)

Transport:

Richter, Sieber (2002)

Higher orders:

Heusler et al. (2004); Müller et al. (2004)

Other universality classes:

Heusler (2001); Nagao, Saito (2003); Bolte, Harrison(2003)

Shot noise:

Schanz, Puhlmann, Geisel (2003)