Universal equidistribution in wave propagation on surfaces of negative curvature

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The system

Let $(\mathcal{M}, g)$ real analytic Riemannian manifold of **negative curvature** $\boldsymbol{\kappa}$, $\dim \mathcal{M} = 2$, $\mathrm{vol}(\mathcal{M}) < \infty$

Δ_g Laplace Beltrami operator on $L^2(\mathcal{M})$

are interested in the solutions of

$$\mathrm{i}\hbar\partial_t \psi(t) = -\hbar^2 \Delta_g \psi(t)$$

for $t \to \infty$, $\hbar \to 0$, with Lagrangian states ψ_0 as initial conditions, $\psi(t) = \mathcal{U}(t)\psi_0$.

$$\mathcal{M} = \mathbf{X}/\mathbf{\Gamma}$$

Γ fundamental group of $\mathcal{M}$, $X = \mathbb{R}^2$ universal cover of $\mathcal{M}$. For $\psi : X \to \mathbb{C}$ define $\psi_\Gamma : \mathcal{M} \to \mathbb{C}$ by

$$\boldsymbol{\psi}_{\mathbf{\Gamma}} = \sum_{\gamma \in \mathbf{\Gamma}} \psi \circ \gamma$$

convergent if $|\psi(x)| \leq C\mathrm{e}^{-\alpha d(x_0, x)}$ for some $x_0 \in X$, $C > 0$ and $\alpha > |\kappa|_\infty$.

main results

Theorem 1. [Equidistribution for large times] *Assume ψ_0 is a Lagrangian state on X satisfying the previous conditions, $\|\psi_{0\Gamma}\|_{L^2(\mathcal{M})} = 1$, and ρ is an analytic function on $\mathcal{M}$. Then for any $\varepsilon > 0$*

$$\lim_{t\to\infty,\hbar\to 0} \int_{\mathcal{M}} |\mathcal{U}(t)\psi_{0\Gamma}|^2 \rho \, \mathrm{d}\nu_g - \frac{1}{\mathrm{vol}(\mathcal{M})} \int_{\mathcal{M}} \rho \, \mathrm{d}\nu_g$$

where the limit is taken such that $t \leq 1/\hbar^{1-\epsilon}$

Theorem 2. [Central limit theorem] *The value distribution of $\mathcal{U}(t)\psi_{0\Gamma}$ becomes Gaussian in the limit $t \to \infty$, $\hbar \to 0$, $t \leq 1/\hbar^{1-\epsilon}$.*

Remarks

- universality, the system forgets its initial state!
- Theorem 1 valid for $A \in \Psi^0(\mathcal{M})$ instead of ρ, extension of previous result (R.S. 04) beyond Ehrenfest time.
- if $\psi_0(x) = a((x - x_0)/\hbar^{\alpha}) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(x)}$, $0 \leq \alpha \leq 1/2$, then the same results hold $C\alpha \ln(1/\hbar) \leq t \leq 1/\hbar^{1-\alpha-\varepsilon}$

chaos and mixing

Let $\Phi^t : S^*\mathcal{M} \to S^*\mathcal{M}$ denote the geodesic flow, $\mathrm{d}\mu$ the normalized Liouville measure on $S^*\mathcal{M}$, and assume $\mathrm{vol}(\mathcal{M}) < \infty$. Then we have

( ) **mixing** (Dolgopyat 98), $a, \rho \in C^1(S^*\mathcal{M})$ with $\int a \,\mathrm{d}\mu = 1$, then

$$\int_{S^*\mathcal{M}} a\rho \circ \Phi^t \,\mathrm{d}\mu \quad \int \rho \,\mathrm{d}\mu + O\big(|a|_{C^1}|\rho|_{C^1}\mathrm{e}^{-\alpha t}\big)$$

(ii) mixing for densities concentrated on submanifolds (Eskin McMullen 93, Sinai 95, Chernov 97, RS 04). Assume Λ_φ satisfies the transversality condition, and let a be a C^1 density on Λ_φ, then

$$\int_{\Lambda_\varphi} \rho \circ \Phi^t a = \int_{\Lambda_\varphi} a \int_{S^*\mathcal{M}} \rho \,\mathrm{d}\mu + O\big(|a|_{C^1}|\rho|_{C^1}\mathrm{e}^{-\alpha' t}\big)$$

curvature

$$\partial_r^2 f(r,s) + (\partial_r f(r,s))^2 = -\kappa(r,s)$$

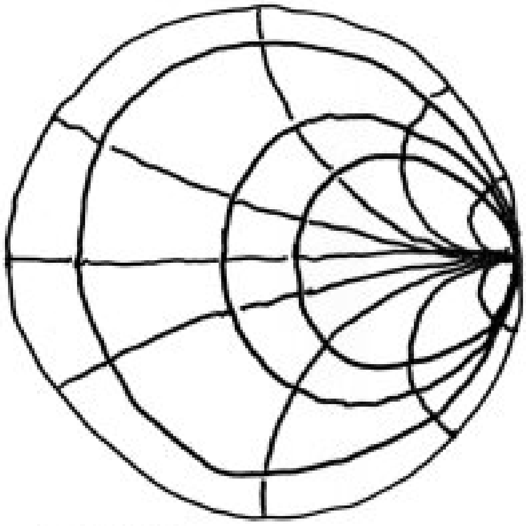
comparison theorem

$$\sqrt{|\kappa|_{min}}\, r + C_1 \leq f(r,s) \leq \sqrt{|\kappa|_{max}}\, r + C_2 \ , \quad r \geq 0$$

Examples: constant curvature $\kappa = -1$

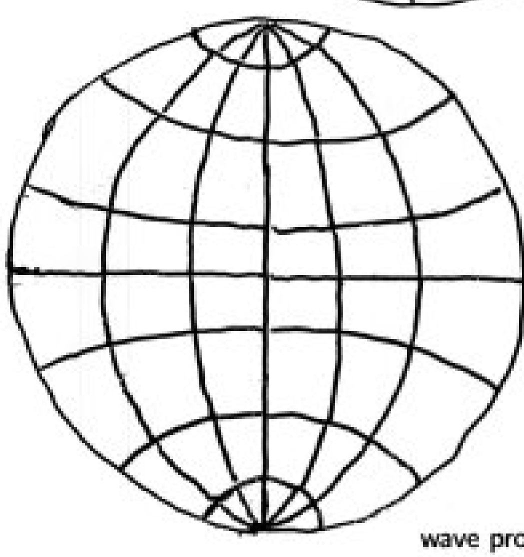
- S horocycle

$f(r,s) = r$
$\Delta_g \quad \partial_r^2 + \partial_r + \mathrm{e}^{-2r}\partial_s^2$
$\mathrm{d}\nu_g \quad \mathrm{e}^r \mathrm{d}r\mathrm{d}s$



- S geodesic

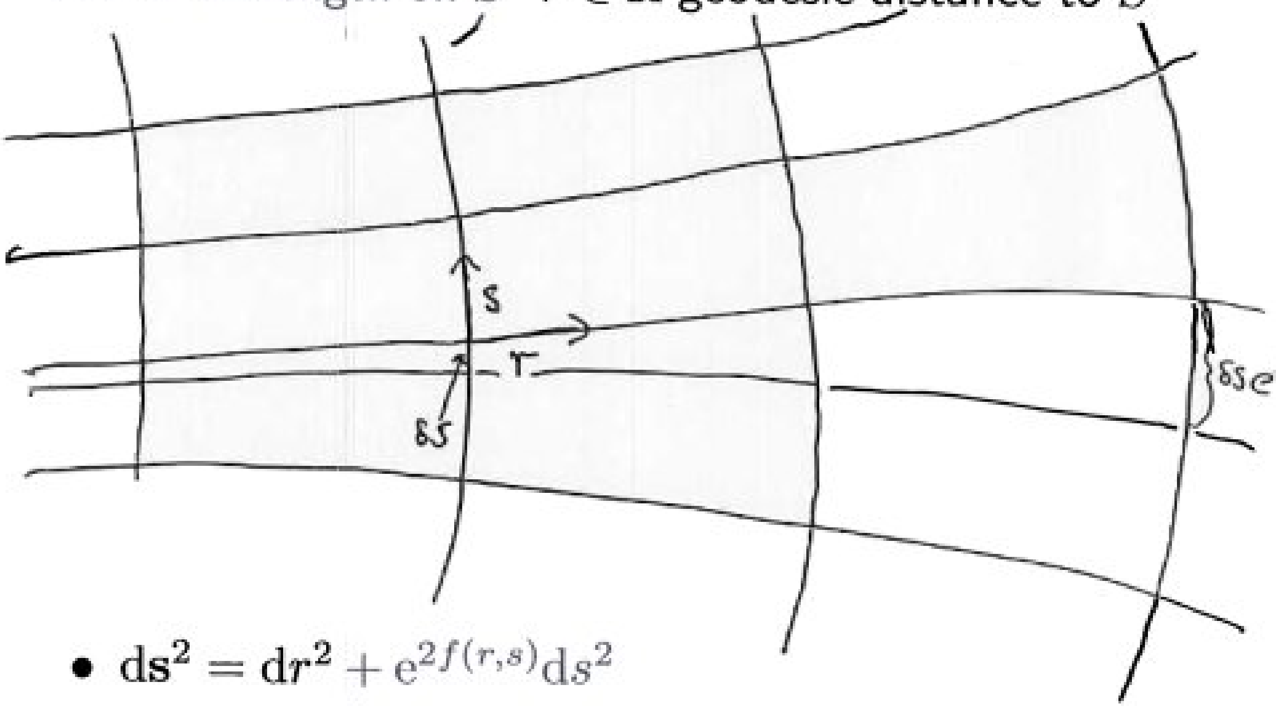
$f(r,s) = \ln\cosh(r)$
$\mathrm{e}^{2f(r,s)} = (\cosh(r))^2$
$f(r) \sim |r| \ r \to \pm\infty$



coordinates on X

$S \subset X$ wavefront, i.e., $\varphi|_S = const$, $\dim S = 1$, S connected $\sim \mathbb{R}$. Choose Fermi coordinates around S:

$s \in \mathbb{R}$ arclength on S $r \in \mathbb{R}$ geodesic distance to S



- $\mathbf{ds^2} = \mathrm{d}r^2 + \mathrm{e}^{2f(r,s)}\mathrm{d}s^2$
- $\varphi(r,s) = r$
- $\Delta_g = \partial_r^2 + \partial_r f(r,s)\partial_r + \mathrm{e}^{-2f(r,s)}\big(\partial_r^2 - \partial_s f(r,s)\partial_s\big)$
- $\mathrm{d}\nu_g = \mathrm{e}^{f(r,s)}\mathrm{d}s\mathrm{d}r$

time evolution on

from now on only: $f(r,s) = r$

Set

$$\psi_0(r,s) = b(r,s)\mathrm{e}^{-r/2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}r}$$

$$\psi(t;r,s) = b(r-2t,s)\mathrm{e}^{-r/2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}r}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}t}$$

Lemma.

$$\mathcal{U}(t)\psi_0 = \psi(t) - \mathrm{i}\hbar\int_0^t \mathcal{U}(t-t')R(t')\mathrm{d}t'$$

with $R(t;r,s) = \big[\partial_r^2 b(r-2t,s) \quad \frac{1}{4}b(r \quad 2t,s) + \mathrm{e}^{-2r}\partial_s^2 b(r-2t,s)\big]\mathrm{e}^{-r/2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}r}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}t}$.

Since $\|R(t)\|_{L^2(X)} \leq C$ for $t \geq 0$,

$$\|\mathcal{U}(t)\psi_0 - \psi(t)\|_{L^2(X)} \leq C\hbar t\ , \quad t > 0$$

Remark: for $t \sim 1/\hbar$ dispersion in r sets in.

projecting to $\mathcal{M}$

$$\mathcal{U}(t)\psi_{0\Gamma} = \psi(t)_\Gamma - \mathrm{i}\hbar\int_0^t \mathcal{U}(t-t')R(t')_\Gamma \mathrm{d}t'$$

naive estimates: $|\psi(t)_\Gamma|,\ R(t')_\Gamma| \leq C\mathrm{e}^t$

$\rho\circ\gamma = \rho$ for all $\gamma \in \Gamma$ and set $\varphi(r,s) = r$ $= \varphi\circ\gamma$

$$\langle\psi(t)_\Gamma, \rho\psi(t)_\Gamma\rangle_{\mathcal{M}} \quad \sum_{\gamma\in\Gamma}\langle\psi(t)\circ\gamma, \rho\psi(t)\rangle_X$$

$$\sum_{|\varphi'_\gamma-\varphi'|<\delta}\langle\psi(t)\circ\gamma, \rho\psi(t)\rangle_X$$ "near diagonal"

$$+\sum_{|\varphi'_\gamma-\varphi'|\geq\delta}\langle\psi(t)\circ\gamma, \rho\psi(t)\rangle_X$$ "off diagonal"

$$\langle\psi(t)\circ\gamma, \rho\psi(t)\rangle_X = \int_X c_\gamma(t,r,s)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}[\varphi-\varphi\circ\gamma]}\,\mathrm{d}\nu_g$$

Gevrey functions

$b \in G^s(\mathbb{R}^2)$, $s \geq 1$, if there are C, R such that

$$|\partial^\alpha b| \leq C R^{|\alpha|} (|\alpha|!)^s$$

for all $\alpha \in \mathbb{N}^2$

- $G^1(\mathbb{R}^2)$ real analytic
- $G_0^s(\mathbb{R}^2) := G^s(\mathbb{R}^2) \cap C_0^\infty(\mathbb{R}^2) \neq \emptyset$ if $s > 1$
- if $b \in G_0^s(\mathbb{R}^2)$ then $|\hat{b}(\xi)| \leq C\mathrm{e}^{-\beta|\xi|^{1/s}}$
- invariant under multiplication and composition with analytic maps

Application: Assume $b \in G_0^s(\mathbb{R}^2)$, $|\varphi'_\gamma - \varphi'| \geq \delta$ then $\langle \psi(t) \circ \gamma, \rho\psi(t) \rangle_X \leq C\mathrm{e}^{-\beta/\hbar^{1/s}}$

$$\left| \sum_{|\varphi'_\gamma - \varphi'| \geq \delta} \langle \psi(t) \circ \gamma, \rho\psi(t) \rangle_X \right| < C\mathrm{e}^{-\beta/\hbar^{1/s}} \mathrm{c}^t$$

near diagonal terms

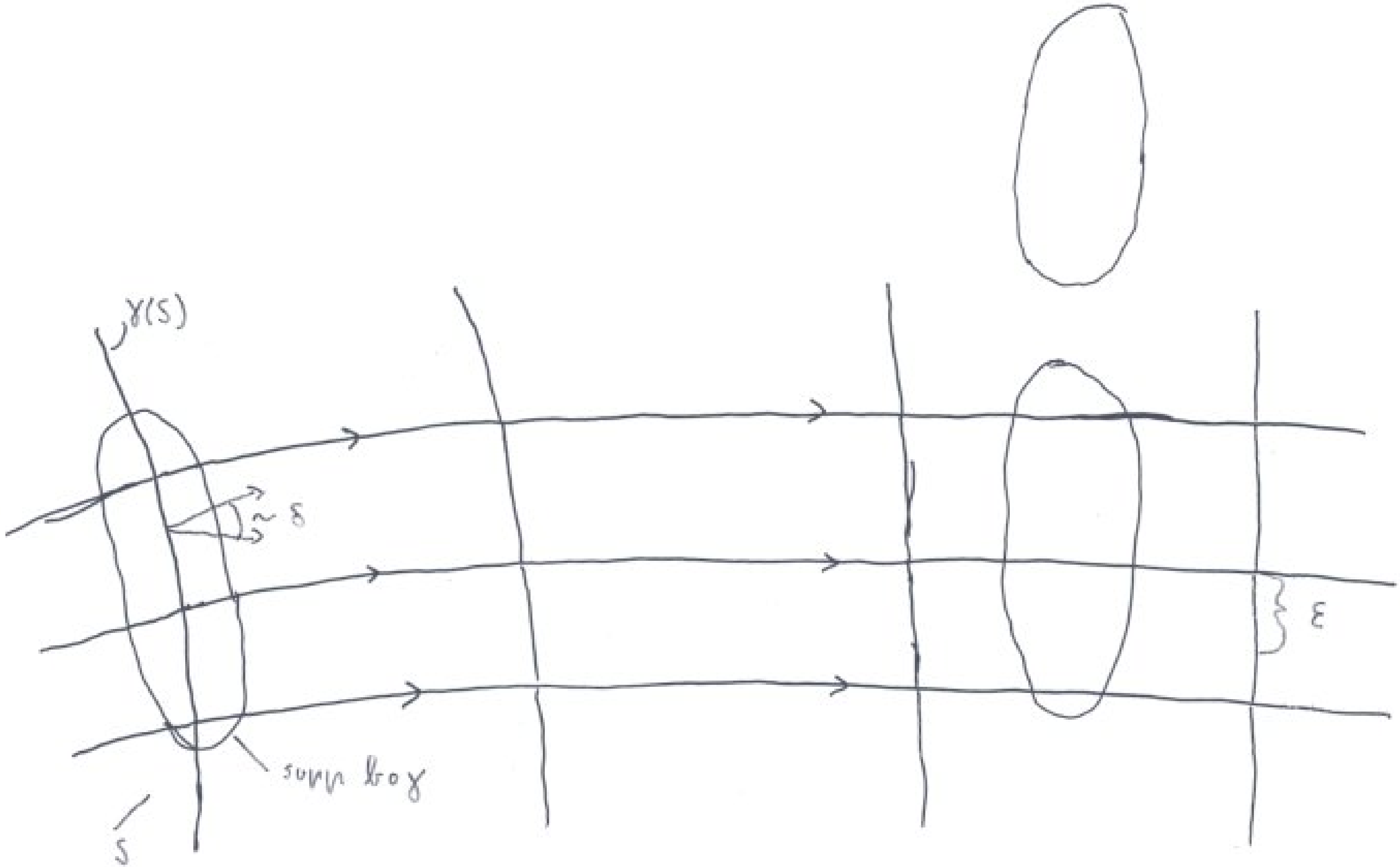
Lemma. $(\xi_r, \xi_s, r, s) \in S^*\mathcal{M}$, *for all* $K \subset \mathbb{R}^2$, $\varepsilon > 0$ *exists* $\delta > 0$ *such that for* $|\xi_s| \leq \delta$, $1 - \xi_r \leq \delta$

$$r(t) - r \quad 2t| \leq \varepsilon \ , \quad |s(t) - s| \leq \varepsilon$$

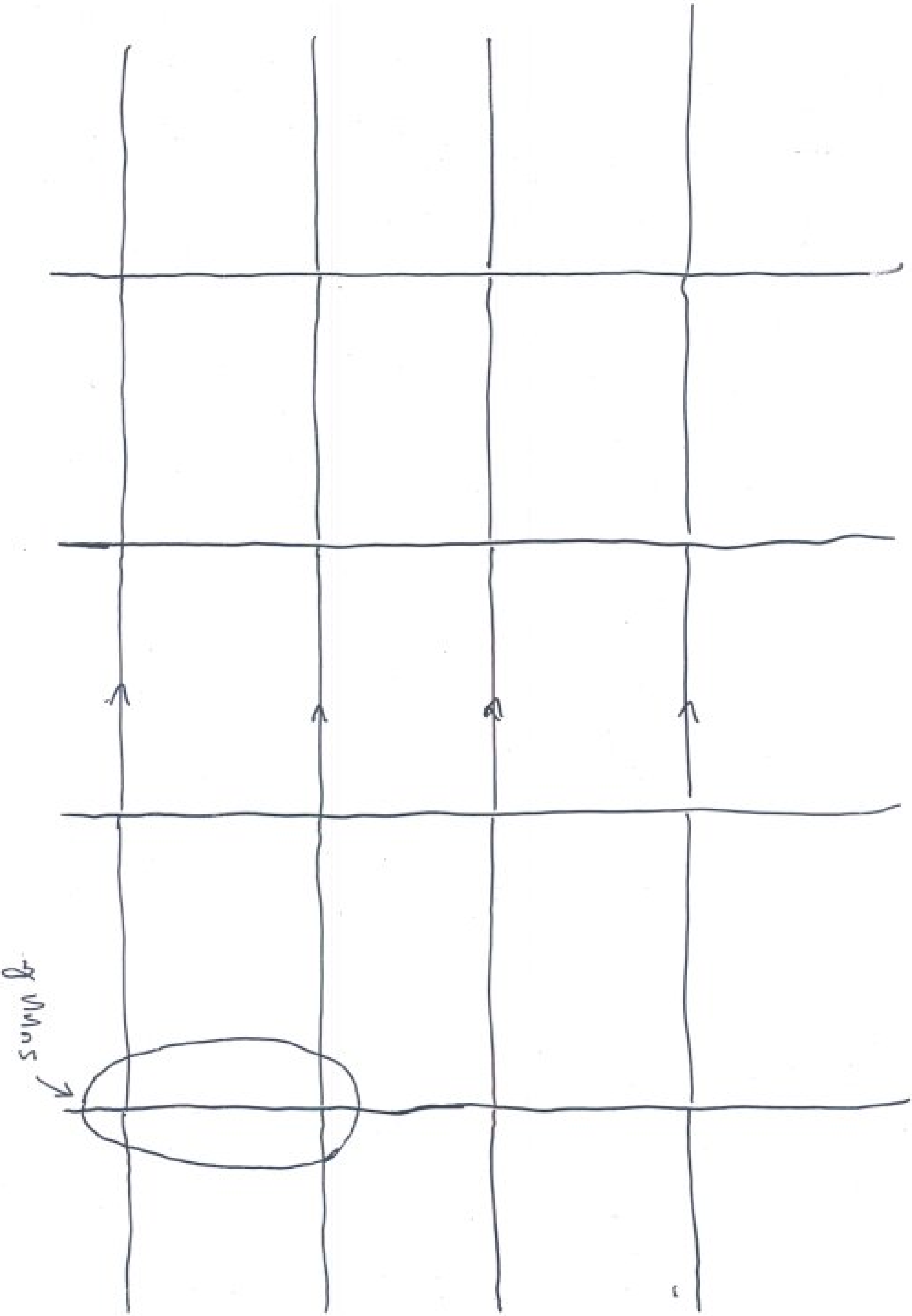
for all $(r, s) \in K$ *and* $t \geq 0$

Application: still $b \in G_0^s(\mathbb{R}^2)$, and enough, $|\varphi'_\gamma \quad \varphi'| < \delta$, then $\psi(t)$) $\gamma \neq id$. So

$$\sum_{|\varphi'_\gamma - \varphi'| < \delta} \langle \psi(t) \circ \gamma, \rho\psi(t) \rangle_X = \langle \psi(t), \rho\psi(t) \rangle_X$$



$\gamma(S)$
δ
ε
S



Finally

Proposition 1. $\psi(t;r,s) = b(r-2t,s)\mathrm{e}^{-r/2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}r}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}t}$ $b \in G_0^s(\mathbb{R}^2)$, $\operatorname{supp} b$ *sufficiently small,* $\psi_0 = \psi(0)$, *then*

$$\langle \psi(t)_\Gamma, \rho\,\psi(t)_\Gamma\rangle_X \quad \iint |b(r,s)|^2 \rho(r+2t,s)\,\mathrm{d}r\mathrm{d}s + O\big(\mathrm{e}^{t-\beta/\hbar^{1/s}}\big)$$

and

$$\|\mathcal{U}(t)\psi_{0\Gamma} - \psi(t)_\Gamma\|_{L^2(\mathcal{M})} = O(\hbar t) + O\big(\mathrm{e}^{t-\beta/\hbar^{1/s}}\big)$$

mixing:

$$\iint |b(r,s)|^2 \rho(r+2t,s)\,\mathrm{d}r\mathrm{d}s$$

$$\iint |b(r,s)|^2\,\mathrm{d}r\mathrm{d}s \,\frac{1}{\mathrm{vol}(\mathcal{M})}\int_{\mathcal{M}} \rho\,\mathrm{d}\nu_g + O(\mathrm{e}^{-\alpha' t})$$

Remark: $\iint |b(r,s)|^2\,\mathrm{d}r\mathrm{d}s = \|\psi_{0\Gamma}\|_{L^2(\mathcal{M})}$

CLT

We use the method of moments. By a combination of the previous methods and mixing one can show for $t \leq C/\hbar^{1/s}$

$$\lim_{t\to\infty \hbar\to 0} \int \left(\psi(t)^*_\Gamma\right)^m \left(\psi(t)_\Gamma\right)^n \, \mathrm{d}\nu_g = \delta_{n,m} n! ||\psi_0||^{2n}$$

Summary and outlook

Results:

- time evolution beyond log-time scale
- universality associated with classical chaos implies universality in quantum time evolution (for a certain class of initial states)
- the system "forgets" its initial state and tends to a universal equidistribution with a central limit theorem valid for the value distribution.

Open problems:

- what happens for more general initial states?
- what happens for $t \sim 1/\hbar$, arithmetic vs. non-arithmetic?
- higher dimensions, what are the correct time scales?

Lagrangian states

$$\psi(x) := a(x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(x)}$$

a, φ analytic on X, φ real.

$$\Lambda_\varphi := \{(\varphi'(x), x)\; x \in X\}$$
$$\Lambda_\varphi \subset S^*X := \{(\xi, x) \in T^*X\, ;\, |\xi|_{g(x)} = 1\}$$

Transversality condition on φ: Λ_φ is transversal to the stable foliation of S^*X: For any (ξ, x), $(\eta, y) \in \Lambda_\varphi$, $(\xi, x) \neq (\eta, y)$, the geodesics through (ξ, x), (η, y) seperate exponentially for $t \to \infty$.

Example: plane waves. $\kappa = -1$, $X \sim D = \{z \in \mathbb{C}; |z| < 1\}$, $b \in \partial D$, $(z, b) :=$ *distance from z to horocycle through $0, b$.* Then $\varphi = (z, b)$ satisfies the transversality condition, whereas $\varphi = -(z, b)$ does not!