On the distribution of matrix elements for the desymmetrized quantum cat map

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Cat maps

- Discrete time dynamical system, “toy model” for understanding Quantum Chaos.

- Phase space: Two dimensional torus, $T^2 = \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$.

- Dynamics given by $A \in SL_2(\mathbf{Z})$ acting on T^2.

- If $|\,\text{trace}(A)| > 2$ get classical chaos.

Quantization

- Planck's constant: $h = 1/N$, N integer tending to ∞.

- State space: $\mathcal{H}_N = L^2(\mathbf{Z}/N\mathbf{Z})$. (Finite dimensional!)

- Algebra of observables acting on $L^2(\mathbf{Z}/N\mathbf{Z})$:

  $$T_N((n_1, n_2))\psi(Q) = e^{\frac{\pi i n_1 n_2}{N}} e^{\frac{2\pi i n_2 Q}{N}} \psi(Q+n_1)$$

- Given $f \in C^\infty(T^2)$, let

  $$Op_N(f) = \sum_{n \in \mathbf{Z}^2} \widehat{f}(n) T_N(n)$$

- Quantum propagator: Unitary map $U_N(A)$ acting on state space $L^2(\mathbf{Z}/N\mathbf{Z})$.

Properties of quantization

- Exact Egorov:

$$U_N(A)^* Op_N(f) U_N(A) = Op_N(f \circ A)$$

- $U_N(A)$ defined up to scalar multiple.

- $U_N(A)$ only depends on $A \mod 2N$.

- On certain subgroup $\Gamma(4) \subset SL_2(\mathbf{Z})$, can define U_N so that

$$U_N(AB) = U_N(A) U_N(B) \quad \text{if } A, B \in \Gamma(4).$$

Behaviour of matrix elements

- Special basis: eigenfunctions of propagator

$$U_N(A)\psi_j = \lambda_j \psi_j, \quad i = 1, \ldots, N$$

- Study (diagonal) matrix elements of observables w.r.t. this basis:

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle$$

- Classical dynamics ergodic ⇒ quantum ergodicity:

  Theorem (Bouzouina-De Bievre, Colin de Verdière, Schnirelman, Zelditch). *For all N there is $S_N \subset \{1, 2, \ldots, N\}$ with $|S_N|/N \to 1$, so that*

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle = \int_{T^2} f + o(1)$$

  for $j \in S(N)$ as $N \to \infty$.

  Theorem follows from variance estimate: as $N \to \infty$,

$$\frac{1}{N} \sum_{j=1}^{N} |\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle - \int_{T^2} f|^2 \to 0$$

Quantum symmetries

- Quantum propagator $U_N(A)$ can have large spectral degeneracies, as big as $N/\log N$.

- In fact, scarring can happen because of these large degeneracies!!! (Recent result by De Biévre, Faure, Nonnenmacher.)

- However, possible to *desymmetrize*: there exists commuting family $\mathcal{C}_A(N)$ of unitary operators acting on $\mathcal{H}_N$ that commute with $U_N(A)$

- No classical analog of these "Hecke symmetries".

- Idea: Take $U_N(B)$ for $B \in SL_2(\mathbf{Z}/2N\mathbf{Z})$ s.t. $AB = BA \mod 2N$. Then

$$U_N(B)U_N(A) = U_N(BA) = U_N(AB) =$$

$$= U_N(A)U_N(B)$$

- Define *Hecke eigenfunctions* as joint eigenfunctions of all $U \in \mathcal{C}_A(N)$

- Still degeneracies, but much smaller.

- Get strong form of quantum ergodicity:

  Theorem (K., Rudnick). *Hecke eigenfunctions are uniformly distributed:*

$$\langle \mathrm{Op}_N(f)\psi_j, \psi_j\rangle \to \int_{T^2} f(x)\, dx$$

  for **all** *Hecke eigenfunctions ψ_j as $N \to \infty$.*

Rate of convergence

- Theorem based on fourth moment estimate

$$\sum_j \left| \langle \mathsf{Op}_N(f)\psi_j, \psi_j\rangle - \int_{T^2} f(x)\,dx \right|^4 \ll N^{-1+\epsilon}$$

- Dropping terms gives

$$\langle \mathsf{Op}_N(f)\psi_j, \psi_j\rangle - \int_{T^2} f(x)\,dx \ll N^{-1/4+\epsilon}.$$

- Expected rate of decay:

$$\langle \mathsf{Op}_N(f)\psi_j, \psi_j\rangle - \int_{T^2} f(x)\,dx \ll N^{-1/2+\epsilon}.$$

- Correct rate now known for primes:

  Theorem (Degli Esposti, Graffi, Isola). *If N is split prime, then*

  $$\langle \mathsf{Op}_N(f)\psi_j, \psi_j\rangle - \int_{T^2} f(x)\, dx \ll N^{-1/2}.$$

  Theorem (Gurevich, Hadani). *If N is inert prime, then*

  $$\langle \mathsf{Op}_N(f)\psi_j, \psi_j\rangle - \int_{T^2} f(x)\, dx \ll N^{-1/2}.$$

Fluctuations

- Natural question: fluctuations around limit as $N \to \infty$ along primes?

- I.e., study normalized fluctuations:

  $$F_j^{(N)} := \sqrt{N} \cdot \left(\langle \mathrm{Op}_N(f)\psi_j, \psi_j \rangle - \int_{\mathbf{T}^2} f(x)dx \right)$$

- For generic chaotic systems, Feingold-Peres, Eckhardt-Fishman-Keating-Agam-Main-Müller and de Corvallo-Keating-Robbins gives following prediction: The distribution is Gaussian with mean zero and variance

  $$\sim V_{\text{generic}}(f) := \sum_{t=-\infty}^{\infty} \langle f_0 \circ A^t, f_0 \rangle$$

  where $f_0 = f - \int_{\mathbf{T}^2} f(y)dy$.

Fluctuations for catmaps

- Canonical basis of observables given by exponentials.

- If $f(x,y) = e^{2\pi i(xn_1+yn_2)}$ then
$$F_j^{(N)} = \sqrt{N} \cdot \left\langle T_N((n_1,n_2))\psi_j, \psi_j \right\rangle$$

- For *split* primes, can interpret
$$\left\langle T_N((n_1,n_2))\psi_j, \psi_j \right\rangle$$
as one variable exponential sum. Suspect Sato Tate distribution (aka semicircle law).

- Numerics confirm suspicion.

General observables:

- If $m = nA^k$, Egorov gives

$$\left\langle T_N(n)\psi_j, \psi_j \right\rangle = \left\langle T_N(m)\psi_j, \psi_j \right\rangle, \quad j = 1, 2, \ldots, N$$

- Matrix elements corresponding to Fourier coefficients in same A-orbit are highly dependent. (To be expected!)

- More surprising: Given A, define a binary quadratic form by

$$Q(x, y) = cx^2 + (d - a)xy - by^2, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $Q(m) = Q(n)$ for $m, n \in \mathbf{Z}^2$, then

$$\left\langle T_N(n)\psi_j, \psi_j \right\rangle = \pm \left\langle T_N(m)\psi_j, \psi_j \right\rangle, \quad j = 1, 2, \ldots, N$$

- Now, $m = nA^k \Rightarrow Q(m) = Q(n)$, but converse is not true.

- Reason: "symmetries visible modulo N, but not over $\mathbf{Z}$". (Similar to Hecke symmetries.)

- Other than this, matrix elements corresponding to different Fourier coefficients should be independent.

- Given observable $f \in C^\infty(\mathbf{T}^2)$, set

$$f^\#(k) := \sum_{\substack{n=(n_1,n_2)\in\mathbf{Z}^2 \\ Q(n)=k}} (-1)^{n_1 n_2} \widehat{f}(n)$$

  Conjecture. *As $N \to \infty$ through primes, the limiting distribution of the normalized matrix elements $F_j^{(N)}$ is that of the random variable*

$$X_f := \sum_{k\neq 0} f^\#(k)\,\mathrm{tr}(U_k)$$

  where $\{U_k\}$ are independent random matrices in $SU(2)$, w.r.t. Haar measure.

Evidence

- Second and fourth moment agrees with conjecture:

  Theorem. *As $N \to \infty$ through primes,*

  $$\frac{1}{N} \sum_{j=1}^{N} |F_j^{(N)}|^{2k} \to \mathbf{E}(X_f)^{2k}$$

  for $k = 1, 2$.

- More numerics. (Confirms independence!)

- For *split* primes, matrix elements are given by one-variable character sums parametrized by multiplicative characters. Recent result:

  Theorem (Katz). *The conjecture is true for split primes.*

Comparison with generic mixing system

- Expect variance of diagonal matrix elements to be:

$$V_{\text{generic}} = \sum_{t=-\infty}^{\infty} \int_{\mathbf{T}^2} f_0(x)\overline{f_0(A^t x)}dx$$

where $f_0 = f - \int_{\mathbf{T}^2} f(y)dy$.

- In terms of Fourier expansion, get:

$$V_{\text{generic}} = \sum_{t=-\infty}^{\infty} \sum_{n\in\mathbf{Z}^2-\{0\}} \widehat{f}(n)\overline{\widehat{f}(nA^t)}$$

$$= \sum_{m\in(\mathbf{Z}^2-\{0\})/\langle A\rangle} \left| \sum_{n\in m\langle A\rangle} \widehat{f}(n) \right|^2$$

- Now, $(n_1, n_2) = (m_1, m_2)A^k \Rightarrow (-1)^{n_1 n_2} = (-1)^{m_1 m_2}$, so

$$V_{\text{generic}} = \sum_{m \in (\mathbf{Z}^2 - \{0\})/\langle A \rangle} \left| \sum_{n \in m\langle A \rangle} (-1)^{n_1 n_2} \widehat{f}(n) \right|^2$$

- For cat map:

$$V_{\text{cat}} = \sum_{\substack{0 \neq m,n \in \mathbf{Z}^2 \\ Q(m)=Q(n)}} (-1)^{n_1 n_2 + m_1 m_2} \widehat{f}(m) \overline{\widehat{f}(n)}$$

$$= \sum_{k \neq 0} \left| \sum_{n \in \mathbf{Z}^2 : Q(n) = k} (-1)^{n_1 n_2} \widehat{f}(n) \right|^2$$

- Answers would agree if A acts transitively on hyperbolas $\{n \in \mathbf{Z}^2 : Q(n) = k\}$.

- Recall: $m = nA^k \Rightarrow Q(m) = Q(n)$, but converse is not true.

Higher dimensional analogue of cat maps

- Classical dynamics: $A \in Sp_4(\mathbf{Z})$ acting on $\mathbf{T}^4 = \mathbf{R}^4/\mathbf{Z}^4$.

- Quantize using Weil representation.

Theorem (Kelmer). *If the characteristic polynomial P_A of A is irreducible over $\mathbf{Q}$, then*

$$\frac{1}{N^2}\sum_{j=1}^{N^2}\left|\langle Op(f)\psi_j,\psi_j\rangle - \int_{\mathbf{T}^4} f\right|^2 \sim c_f \cdot \frac{1}{N^2}$$

If P_A is reducible over $\mathbf{Q}$, then

$$\frac{1}{N^2}\sum_{j=1}^{N^2}\left|\langle Op(f)\psi_j,\psi_j\rangle - \int_{\mathbf{T}^4} f\right|^2 \sim \tilde{c}_f \cdot \frac{1}{N}$$

Note: wrong order of magnitude if P_A reducible!!