Weyl's Law for Heisenberg Manifolds

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1. Heisenberg Group
2. Heisenberg manifolds
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4. Spectrum of Heisenberg manifolds
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Heisenberg algebra:

$$\mathfrak{h}_n = \langle X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z \rangle$$

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0$$

$$[X_i, Y_j] = \delta_{ij} Z$$

Heisenberg Group:

$$H_n = \left\{ \begin{bmatrix} 1 & \mathbf{x} & z \\ 0 & I_n & {}^t\mathbf{y} \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{x}, \mathbf{y} \in \mathbf{R}^n, z \in \mathbf{R} \right\}$$

$$X(\mathbf{x}, \mathbf{y}, z) = \begin{bmatrix} 0 & \mathbf{x} & z \\ 0 & 0 & {}^t\mathbf{y} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{h}_n = \{X(\mathbf{x}, \mathbf{y}, z)\} \subset \mathfrak{gl}(n+2, \mathbf{R})$$

$$\{X(\mathbf{x}, \mathbf{y}, 0), \mathbf{x}, \mathbf{y} \in \mathbf{R}^n\} \equiv \mathbf{R}^{2n}$$

Center, derived subalgebra: $\mathfrak{z}_n = \{X(0, 0, z)\}$

$$\mathfrak{h}_n = \mathbf{R}^{2n} + \mathfrak{z}_n$$

Heisenberg manifolds

$$(\Gamma \setminus H_n, g)$$

Γ uniform discrete, g left-invariant metric

$$\begin{array}{ccc} S^1 & \hookrightarrow & \Gamma \setminus H_n \\ & & \downarrow \\ & & T^{2n} \end{array}$$

Circle bundle over a torus

Take $\mathbf{r} \in \mathbf{Z}_+^n$, $r_j | r_{j+1}$. Define

$$\Gamma_\mathbf{r} = \left\{ \begin{bmatrix} 1 & \mathbf{x} & z \\ 0 & I_n & {}^t\mathbf{y} \\ 0 & 0 & 1 \end{bmatrix}, x_i \in r_i\mathbf{Z}, \mathbf{y} \in \mathbf{Z}^n, z \in \mathbf{Z} \right\}$$

Theorem (Gordon-Wilson) (a) $\exists \mathbf{r}$:

$$(\Gamma \setminus H_n, g) \sim (\Gamma_\mathbf{r} \setminus H_n, \tilde{g})$$

(b) $\exists \phi \in \mathrm{Inn}(H_n) : \mathfrak{h}_n = \mathbf{R}^{2n} \oplus \mathfrak{z}_n$, rel. $\phi^*(g)$

$$\phi^*(g) = \begin{bmatrix} h_{2n \times 2n} & 0 \\ 0 & g_{2n+1} \end{bmatrix}$$

Why do we care?

1. H_n model for CR manifolds (Folland, Stein, Beals, Greiner)

2. Integrable: Butler, J. Geom. Phys. 36(2000)

One integral is NOT analytic function

Taimanov: Analytic integrals constrain $\pi_1(M)$

3. Isospectral manifolds: Gordon, Wilson, Webb, Gornet, Pesce

$n = 1$, $(\Gamma \setminus H_1, g)$ determined by its spectrum among Heisenberg

$n > 1$, if $r_1 r_2 \cdots r_n = r'_1 r'_2 \cdots r'_n$, continuous families

$$\text{Spec}(\Gamma \setminus H_n, g_t) = \text{Spec}(\Gamma' \setminus H_n, g'_t)$$

Almost Inner Automorphisms exist in abundance.

4. (H_1, g) is a model geometry in classification of 3-manifolds

5. Fourier coefficients of automorphic forms are

$$\int_{\Gamma \cap N \backslash N} \phi(ng)\, dn$$

where $\Gamma \cap N \setminus N$ is Heisenberg manifold.

The spectrum of Heisenberg manifolds

Deninger-Singhof, Colin de Verdière, Gordon-Wilson

Let $L_{\mathbf{r}} = \{X(\mathbf{x}, \mathbf{y}, 0), x_i \in r_i\mathbf{Z}, \mathbf{y} \in \mathbf{Z}^n\}$

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

$\pm i d_j^2$ be the eigenvalues of $h^{-1}J$

Type I:

$$\text{Spec}(L_{\mathbf{r}} \setminus \mathbf{R}^{2n}, h)$$

Type II:

$$\mu(y, t_1, t_2, \ldots, t_n) = \frac{4\pi^2 y^2}{g_{2n+1}} + \sum_{j=1}^{n} 2\pi y d_j^2 (2t_j + 1),$$

$y \in \mathbf{N}$, $t_i \in \mathbf{Z}_+$, with multiplicity $2y^n r_1 r_2 \cdots r_n$.

Heat kernel, determinant of Laplace operator: Furutani, de Gosson J. Geom. Phys. 48(2003)

Question: Are (H_n, g) Quantum Integrable?

Example: On H_1

$$g_0 = \begin{bmatrix} I_2 & 0 \\ 0 & 2\pi \end{bmatrix}$$

2π factors:

$$\frac{1}{2\pi}\mu(y,t) = \Big(y^2 + y(2t+1)\Big) = y(y{+}2t{+}1) = yx$$

with $x > y$, $x \not\equiv y (\mathrm{mod}\ 2)$

Leads to lattice-point counting with weight y below hyperbola $xy = \lambda$ and line $x = y$

Use $\mathbf{Z}^2$ and $L = \{(x,y) \in \mathbf{Z}^2, x \equiv y (\mathrm{mod}) 2\}$

Gauss circle problem:

$$N(\lambda) = \#\{(x,y) \in \mathbf{Z}^2, x^2 + y^2 \leq \lambda\}$$

$$N(\lambda) = \pi\lambda + R(\lambda)$$

$$R(\lambda) = O(\lambda^{1/2})$$

Hardy conjecture:

$$R(\lambda) = O(\lambda^{1/4+\epsilon})$$

Dirichlet divisor problem:

$$N(\lambda) = \#\{(x,y) \in \mathbf{N}^2, xy \leq \lambda\}$$

$$N(\lambda) = \lambda \log \lambda + (2\gamma - 1)\lambda + \Delta(\lambda)$$

$$\Delta(\lambda) = O(\lambda^{1/2})$$

Conjecture:

$$\Delta(\lambda) = O(\lambda^{1/4+\epsilon})$$

Spectral counting function

$$N(\lambda) = \#\{\lambda_j \leq \lambda\}$$

Weyl's law, Hörmander's Theorem:

$$N(\lambda) = c_n \mathrm{vol}(M)\lambda^{n/2} + R(\lambda)$$

with

$$R(\lambda) = O(\lambda^{(n-1)/2})$$

In 3-dim gives $R(\lambda) = O(\lambda)$

Theorem 1: (Chung, P., Toth) For every left-invariant metric g on H_1

$$R(\lambda) = O(\lambda^{34/41})$$

Conjecture 1: $R(\lambda) = O(\lambda^{3/4+\epsilon})$

Remark: For $\mathbf{Z}^3\backslash\mathbf{R}^3$, conj.: $R(\lambda) = O(\lambda^{1/2+\epsilon})$

$$c_n = \frac{1}{(2\pi)^n}\frac{\pi^{n/2}}{\Gamma(n/2+1)}$$

Exponent pair we used $(k,l) = (11/30, 16/30)$

Huxley for Gauss: $R(\lambda) = O(\lambda^{23/73+\epsilon})$

gives for Heisenberg $R(\lambda) = O(\lambda^{119/146+\epsilon})$

Theorem 2: (Chung, P., Toth) If (k, l) is an exponent pair then

$$R(\lambda) = O(\lambda^{(l+2k+1)/(2k+2)})$$

Def. Let $0 \leq k \leq 1/2 \leq l \leq 1$. We call (k, l) an exponent pair if $\forall s > 0$ $\exists P(k, l, s)$ $\forall N > 0$ $\forall t > 0$ and $\forall f(x)$ with

$$f'(x) \sim tx^{-s}, \quad f''(x) \sim -stx^{-s-1}, \ldots$$

$$f^{(P)}(x) \sim (-1)^{P+1} s(s+1) \cdots (s+P-2) tx^{-s-P+1}$$

we have

$$\sum_{N \leq n \leq 2N} e(f(n)) \ll (tN^{-s})^k N^l + t^{-1} N^s$$

Conjecture: (Montgomery) $\forall \epsilon$, $(\epsilon, 1/2 + \epsilon)$ is an exponent pair.

Implies Lindelöf H, Hardy's conjecture

Implies Conjecture 1 for Heisenberg

Evidence:

Theorem 3: (Khosravi) Cramér's formula:

$$\lim_{T\to\infty} \frac{1}{T^{5/2}} \int_0^T R(\lambda)^2 \, d\lambda = c > 0$$

Compare with:

Cramér (1922)

$$\lim_{T\to\infty} \frac{1}{T^{3/2}} \int_0^T R(\lambda)^2 \, d\lambda = \frac{1}{3\pi^2} \sum_n \left(\frac{r(n)}{n^{3/4}} \right)^2$$

for Gauss circle problem,

$$\lim_{T\to\infty} \frac{1}{T^{3/2}} \int_0^T \Delta(\lambda)^2 \, d\lambda = \frac{1}{6\pi^2} \sum_n \left(\frac{d(n)}{n^{3/4}} \right)^2$$

for Dirichlet divisor problem

Average over metrics g_u

Theorem 4: (P., Toth) For a local perturbation g_u, $u \in I$ of g_0

$$\int_I R(\lambda, u)^2 \, du = O(\lambda^{3/2})$$

Kendall (1948) shifts of $\mathbf{Z}^2$

$$\int_0^1 \int_0^1 R(\lambda, \alpha, \beta)^2 \, d\alpha \, d\beta = O(\lambda^{1/2})$$

Theorem 4': (P., Toth) For a local perturbation L_u of the standard lattice $\mathbf{Z}^2$

$$\int_I R(\lambda, u)^2 \, du = O(\lambda^{1/2})$$

Theorem 5: (P., Toth) For fixed g

$$\frac{1}{T} \int_T^{2T} R(\lambda) \, d\lambda \gg T^{3/4}$$

Higher dimensions?

Hörmander: $R(\lambda) = O(\lambda^n)$ on H_n

If d_i^2/d_j^2 are all rational, we call g rational.

Theorem 6: (Khosravi) $n > 1$

(a) Rational case: $R(\lambda) = O(\lambda^{n-7/41})$

(b) Irrational case: $R(\lambda) = O(\lambda^{n-1/4+\epsilon})$ for generic g

Conjecture 2 (Khosravi) $n > 1$ and g rational:

$$R(\lambda) = O(\lambda^{n-1/4+\epsilon})$$

Remark 1: Phase is linear in $n-1$ parameters.

Remark 2: A generic irrational θ satisfies diophantine condition

$$||j\theta|| \gg \frac{1}{j \log^2 j}$$

Hint of proof of Th. 1

Need two term asymptotics for lattice point counting

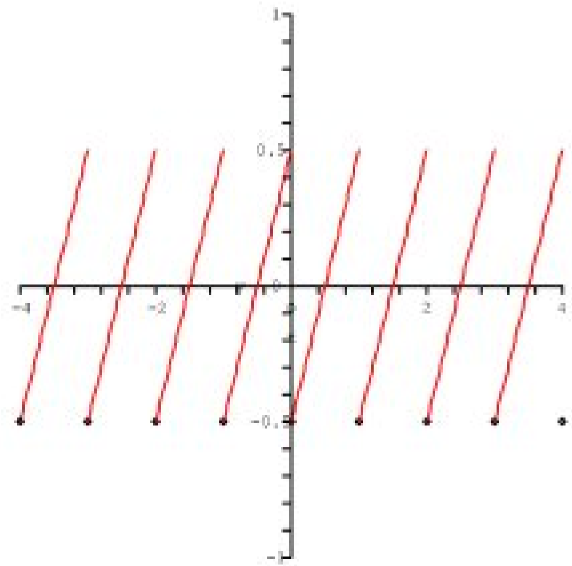
Need to see cancellation with $N_I(\lambda) \sim c\lambda$ for Type I (torus) eigenvalues

For g_0, $x, y \in \mathbf{N}$

$$\sum_{\substack{xy \le \lambda \\ x > \sqrt{\lambda}}} y = \sum_{y \le \sqrt{\lambda}} y \sum_{\sqrt{\lambda} < x \le \lambda/y} 1 = \sum_{y \le \sqrt{\lambda}} y([\lambda/y] - [\sqrt{\lambda}])$$

$$= \sum_{y \le \sqrt{\lambda}} y(\lambda/y - \sqrt{\lambda} - \psi(\lambda/y) + \psi(\sqrt{\lambda}))$$

where $\psi(u) = u - [u] - 1/2$



$$\psi(u) = - \sum_{n \neq 0} \frac{e^{2\pi i n u}}{2\pi i n}$$

Use Euler summation

$$\sum_{n \leq X} n = \frac{X^2}{2} - \psi(X)X + O(1)$$

Reduce to

$$\sum_{y \leq \sqrt{\lambda}} y\psi(\lambda/y)$$

Distribution of $\lambda^{-3/4}R(\lambda)$?

Is it in B^2 Besicovitch class?

Heath-Brown (1992): $\exists f(a)$ for Gauss and $g(a)$ for Dirichlet divisor:

$$\frac{1}{X}\text{measure}\{x \in [1,X], x^{-1/4}R(x) \in I\} \to \int_I f(a)\,da$$

$$\frac{1}{X}\text{measure}\{x \in [1,X], x^{-1/4}\Delta(x) \in I\} \to \int_I g(a)\,da$$

Bleher (1993): Shifted lattices

Kosygin, Minasov, Sinai (1993): Liouville tori