Granular Bosonization
(or Fyodorov meets SUSY)

Newton Institute, Cambridge (July 2, 2004)

Bosonization of Dirac fermions

Functional integral: $Z = \int \exp\left(-\int_{\Sigma} L\, d^2x\right)$

Lagrangian for Dirac fields:

$$L_{\mathrm{F}} = \overline{\psi}\gamma^{\mu}(\mathrm{i}\partial_{\mu} + A_{\mu})\psi + m\overline{\psi}\psi$$

Lagrangian for boson fields:

$$L_{\mathrm{B}} = \frac{1}{8\pi}\partial^{\mu}\varphi\,\partial_{\mu}\varphi + \varepsilon^{\mu\nu}A_{\mu}\partial_{\nu}\varphi + m\cos\varphi$$

Bosonization is the statement $Z_{\mathrm{F}}[A] = Z_{\mathrm{B}}[A]$.
(Note: mass renormalization)

Bosonization rule: $\overline{\psi}_{\uparrow}\psi_{\uparrow} \to \mathrm{e}^{\mathrm{i}\varphi}$, $\overline{\psi}_{\downarrow}\psi_{\downarrow} \to \mathrm{e}^{-\mathrm{i}\varphi}$.

Outline

- Fyodorov's method (2002) for reciprocals of characteristic polynomials of random matrices
- Universal laws for beta = 1, 2, 4
- Existence of extended states in a 3d granular model
- Generalizing Fyodorov's method to the case of characteristic polynomials (fermionic variant): more universal laws
- Extension to ratios of characteristic polynomials (SUSY)

Fyodorov's Method for Reciprocals of Char. Polynomials

Suppose we want to average (determinants)^{-1}

w.r.t. GUE_N measure $\quad d\mu = \mathrm{const} \times \mathrm{e}^{-\mathrm{Tr}H^2/2\lambda^2}\, dH.$

$$\left\langle \prod_{\alpha=1}^{n} \mathrm{Det}^{-1}(E_\alpha + \mathrm{i}\varepsilon - H) \right\rangle_{\mathrm{GUE}_N}$$

$$= \left\langle \prod_{\alpha=1}^{n} \int \mathrm{e}^{\mathrm{i}(\overline{\varphi}_\alpha, (E_\alpha + \mathrm{i}\varepsilon - H)\varphi_\alpha)} \right\rangle_{\mathrm{GUE}_N}$$

$$= \int \mathrm{e}^{\mathrm{i}\Sigma_\alpha (E_\alpha + \mathrm{i}\varepsilon)(\overline{\varphi}_\alpha, \varphi_\alpha)} \;\; \mathrm{e}^{-(\lambda^2/2)\Sigma_{\alpha\beta}(\overline{\varphi}_\alpha, \varphi_\beta)(\overline{\varphi}_\beta, \varphi_\alpha)}$$

$$= \int \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}(E + \mathrm{i}\varepsilon)M} \;\; \mathrm{e}^{-(\lambda^2/2)\mathrm{Tr}M^2}\, \mathrm{Det}^{N-n}(M)\, dM.$$

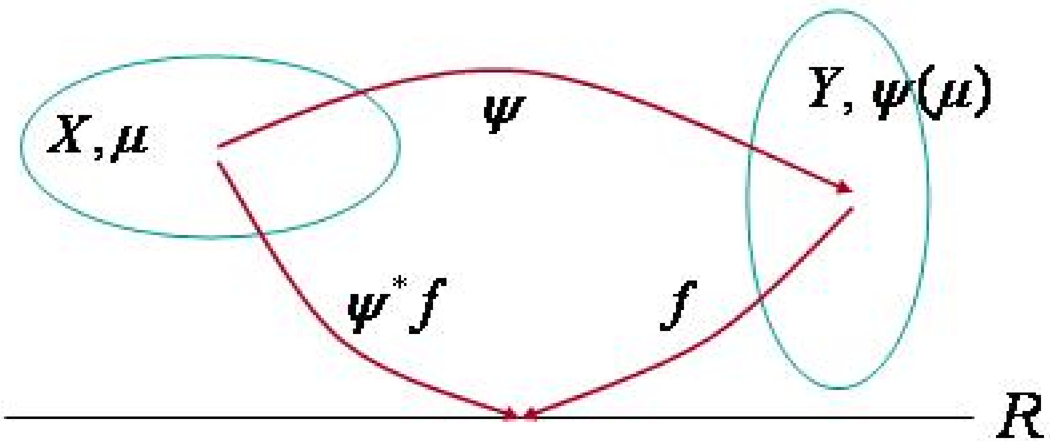
push forward by $\quad M_{\alpha\beta} = (\overline{\varphi}_\alpha, \varphi_\beta) \;\; ; \quad E = \mathrm{diag}(E_1, \ldots, E_n)$

Push Forward

Mapping $\psi : X \to Y$

Pull back of functions: $\psi^* f = f \circ \psi$

Push forward of distributions: $\psi(\mu)[f] = \mu[f^*\psi]$



Application to Fyodorov's Case

$$X := \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^N) \;\to\; \mathrm{Herm}^{\geq 0}(\mathbb{C}^n) =: Y$$

$$\boldsymbol{\varphi} \;\mapsto\; \boldsymbol{\varphi}^* \boldsymbol{\varphi} =: \boldsymbol{M}$$

$$\prod_{i=1}^{N} \prod_{\alpha=1}^{n} d\boldsymbol{\varphi}_{i,\alpha}\, d\overline{\boldsymbol{\varphi}}_{i,\alpha} \;\mapsto\; \mathrm{Det}^{N-n}(\boldsymbol{M})\, d\boldsymbol{M}\,.$$

Normalization:

$$\int_X \mathrm{e}^{-\mathrm{Tr}\varphi^*\varphi} \prod d\boldsymbol{\varphi}\, d\overline{\boldsymbol{\varphi}} = \int_Y \mathrm{e}^{-\mathrm{Tr}M} \mathrm{Det}^{N-n}(\boldsymbol{M})\, d\boldsymbol{M}$$

Note the requirement $N \geq n$.

$\mathrm{GL}(\boldsymbol{n}, \mathbb{C})$ acts by $\boldsymbol{M} \mapsto \boldsymbol{TMT}^*$.

$\mathrm{GL}(\boldsymbol{n}, \mathbb{C})$-invariant measure: $d\mu(\boldsymbol{M}) = \mathrm{Det}^{-n}(\boldsymbol{M})\, d\boldsymbol{M}$.

$$\int F(\boldsymbol{\varphi}^* \boldsymbol{\varphi})\, d\boldsymbol{\varphi}\, d\overline{\boldsymbol{\varphi}} = \int_{M>0} F(\boldsymbol{M})\, \mathrm{Det}^N(\boldsymbol{M})\, d\mu(\boldsymbol{M})$$

"bosonization"

Generalization to Non-Gaussian Distributions

Consider U(N)-invariant measure $d\mu_{V,N}(H) = \mathrm{e}^{-N \operatorname{Tr} V(H)} dH$

with Fourier transform $\left\langle \mathrm{e}^{\mathrm{i} \operatorname{Tr} HK} \right\rangle_{\mu_{V,N}} = \Omega_N(K).$

$$\left\langle \prod_{\alpha=1}^{n} \mathrm{Det}^{-1}(E_\alpha + \mathrm{i}\varepsilon - H) \right\rangle_{\mathrm{GUE}_N}$$

$$= \left\langle \prod_{\alpha=1}^{n} \int \mathrm{e}^{\mathrm{i}(\overline{\varphi}_\alpha, (E_\alpha + \mathrm{i}\varepsilon - H)\varphi_\alpha)} \right\rangle_{\mathrm{GUE}_N}$$

$$K = \sum_\alpha \varphi_\alpha (\overline{\varphi}_\alpha, \cdot) = \varphi\varphi^*$$

$$= \int \mathrm{e}^{\mathrm{i}\sum_\alpha (E_\alpha + \mathrm{i}\varepsilon)(\overline{\varphi}_\alpha, \varphi_\alpha)} \underbrace{\left\langle \mathrm{e}^{\mathrm{i} \operatorname{Tr} HK} \right\rangle_{\mu_{V,N}}}_{= \Omega_N(\varphi\varphi^*) = \widetilde{\Omega}_N(\varphi^*\varphi)}$$

$$= \int \mathrm{e}^{\mathrm{i} N \operatorname{Tr}(E + \mathrm{i}\varepsilon) M} \, \widetilde{\Omega}_N(M) \, \mathrm{Det}^N(M) \, d\mu(M).$$

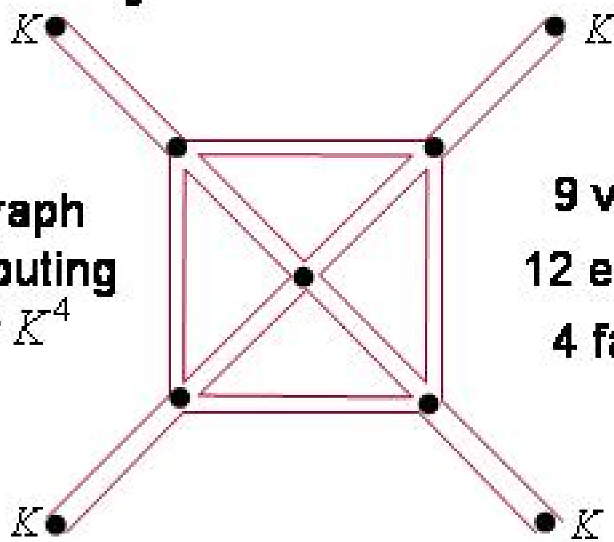
Heuristics from Feynman Graphs

U(N)-invariant probability density for Hermitian $N \times N$ matrices H:

$$d\mu_{V,N}(H) = e^{-N \operatorname{Tr} V(H)} dH .$$

Logarithm of Fourier transform has asymptotic 1/N expansion:

$$-\log \int e^{-N \operatorname{Tr} V(H) + iN \operatorname{Tr} HK} dH = \sum_{\chi = 1, -1, \ldots} N^{\chi} \omega_{\chi}(K).$$



Ex.: graph contributing to $\operatorname{Tr} K^4$

9 vertices: N^9

12 edges: N^{-12}

4 faces: N^4

t'Hooft, Witten, Gross,...

Leading contribution $\omega_1(K)$ comes from summing planar graphs (Euler characteristic $\chi = 1$) .

The Universal Laws

Suppose $-\lim_{N\to\infty} N^{-1} \log \left\langle e^{iN\,\mathrm{Tr}HK} \right\rangle_{\mu_{V,N}} = \omega_1(K)$ exists

and is nonzero. Then the scaling limit

$$F(\xi,\eta) = \lim_{N\to\infty} N^{-p} \left\langle \prod_{a=1}^{n} \mathrm{Det}^{-1}(E + i\varepsilon - H + \xi_a / N) \times \prod_{b=1}^{n} \mathrm{Det}^{-1}(E - i\varepsilon - H + \eta_b / N) \right\rangle_{\mu_{V,N}}$$

exists, and in the bulk of the spectrum is given by

$$F_\beta(\xi,\eta) = \prod_{a,b=1}^{n} \frac{1}{\xi_a - \eta_b} \times f_\beta(\xi,\eta)$$

$$\beta = 2: \quad f_2(\xi,\eta) = \prod_c e^{i\pi(\xi_c - \eta_c)}$$ U(N)-inv.

$$\beta = 1,4: \quad f_\beta(\xi,\eta) = \frac{\mathrm{Det}\big(K_\beta(\xi_a - \eta_b)\big)}{\Delta(\xi)\,\Delta(\eta)}$$

$K_1(t) = \int_1^\infty e^{i\pi t x} x^{-1} dx$ O(N)-inv.

$K_4(t) = \int_1^\infty e^{i\pi t x} x\, dx$ USp(2N)-inv.

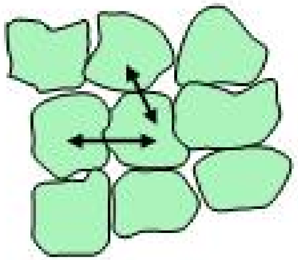
Fyodorov, Keating (2003)

Existence of extended states in a granular model

("Weakly coupled GUEs")

disordered metallic grains

with N electron states



prob. measure $d\boldsymbol{\mu}(H) = c\,|\,\mathrm{Det}(E + \mathrm{i}\boldsymbol{\varepsilon} - H)|^{-2}\, e^{-\Sigma_{\boldsymbol{g}} J^{-1}{}_{\boldsymbol{g}} \mathrm{Tr}\,\Pi_{\boldsymbol{i}} H \Pi_{\boldsymbol{j}} H} dH$

Theorem (T. Spencer, MRZ): In sigma model approximation,

$$\left\langle |\,(E + \mathrm{i}\varepsilon - H)^{-1}(0,0)\,|^{2} \right\rangle_{\mu} \leq \mathrm{const}$$

if $d \geq 3$, $\varepsilon \cdot \mathrm{vol} \geq 1$ and the intergrain coupling is not too small.

This establishes the existence of absolutely continuous spectrum and extended eigenfunctions in this model.

Berezin Integration (the Fermi Integral)

Exterior algebra $\wedge(C^N)$ with generators $\xi_1,\ldots,\xi_N$:

$$\xi_i\xi_j + \xi_j\xi_i = 0.$$

Integration rules $(N=1)$:

$$\int d\xi \cdot 1 = 0, \qquad \int d\xi \cdot \xi = 1.$$

Note $\int d\xi \equiv \partial / \partial\xi$.

Gaussian integral:

$$\int d\psi_1 d\overline{\psi}_1 \ldots d\psi_N d\overline{\psi}_N \exp\left(\sum_{i,j} \overline{\psi}_i A_{ij} \psi_j\right) = \mathrm{Det}\,(A)$$

Generalizing Fyodorov's Method

Bosonic variant:

$M = (M_{\alpha\beta}) = (\overline{\varphi}_\alpha \cdot \varphi_\beta)$ positive Hermitian

$$\int F(\overline{\varphi} \cdot \varphi)\, d\varphi\, d\overline{\varphi} = \int_{M>0} F(M)\, \mathrm{Det}^N(M)\, d\mu(M)$$

with $d\mu(M)$ invariant under $M \to TMT^*$

Fermionic variant:

Scalar products of "anticommuting" vectors: $(\overline{\psi}_\alpha \cdot \psi_\beta)$

$$\int F(\overline{\psi} \cdot \psi)\, d\psi\, d\overline{\psi} = \int_{\mathrm{U}(n)} F(U)\, \mathrm{Det}^{-N}(U)\, d\mu(U)$$

for some Haar measure $d\mu(U)$

The Simplest Example

Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function.

For a single vector ψ with anticommuting entries $\psi_1, \ldots, \psi_N$ consider $F(\overline{\psi} \cdot \psi)$.

$$\int F(\overline{\psi} \cdot \psi)\, d\psi\, d\overline{\psi}$$

$$= \frac{\partial^2}{\partial \psi_1 \partial \overline{\psi}_1} \cdots \frac{\partial^2}{\partial \psi_N \partial \overline{\psi}_N} F(\overline{\psi}_1 \psi_1 + \ldots + \overline{\psi}_N \psi_N)$$

$= F^{(N)}(0)$ (the N-th derivative at the origin)

$$= \oint F(e^{i\theta})\, e^{-iN\theta}\, d\theta / 2\pi.$$

Sketch of proof --- general case

Let $F : \mathrm{End}(\mathbb{C}^n) \to \mathbb{C}$ be an entire function.

For $U_L, U_R \in \mathrm{U}(n)$ define $F_{U_L,U_R}(M) := F(U_L M U_R)$.

Distribution 1: $\mu_1[F] = \int_{\mathrm{U}(n)} F(U)\,\mathrm{Det}^{-N}(U)\,d\mu(U)$

Distribution 2: $\mu_2[F] = \int F(\bar{\psi}\cdot\psi) \prod_{i,\alpha} d\psi_i^\alpha d\bar{\psi}_i^\alpha$

The transformation behavior is the same:

$$\mu_A[F_{U_L,U_R}] = \mathrm{Det}^N(U_L)\,\mathrm{Det}^N(U_R)\,\mu_A[F] \qquad (A = 1,2)$$

$\mu_1 \propto \mu_2$ now follows since the action of the dual pair $\mathrm{U}(N)\times\mathrm{U}(2n)$ on the spinor representation of $\mathrm{Spin}(4nN)$ is multiplicity-free.

Generalization to $\beta = 1$

Physical space $\mathbb{C}^N$ with unitary structure $\langle \cdot , \cdot \rangle$.

Symmetric form $s(v,w) = \langle Tv, w \rangle$.

time reversal operator

Auxiliary space $\mathbb{C}^{2n}$ with alternating form a.

Define a-transpose by $a(Bx, y) = a(x, B^{\mathrm{t}} y)$.

$\psi : \mathbb{C}^{2n} \to \mathbb{C}^N$

$\tilde{\psi} : \mathbb{C}^N \to \mathbb{C}^{2n}$ (canonical adjoint).

$$\int F(\tilde{\psi}\psi)\, d\psi = \int_{G/K} F(UU^{\mathrm{t}})\, \mathrm{Det}^N(U)\, dU_K$$

$G = \mathrm{U}(2n)$, $K = \mathrm{USp}(2n)$.

For $\beta = 4$ exchange $s \leftrightarrow a$, ... $K = \mathrm{SO}(2n)$.

More Universal Laws

Under the same assumption on $\Omega_N(K)$ as before

the scaling limit

$$F(\zeta) = \lim_{N\to\infty} N^p \left\langle \prod_{a=1}^{2n} \mathrm{Det}(H - \zeta_a / N) \right\rangle_{\mu_{V,N}}$$

exists, and in the bulk of the spectrum is given by

$$F_\beta(\zeta) = \Delta(\zeta)^{-1} \sum_{\pi \in S_{2n}} (-1)^{|\pi|} \prod_{a=1}^{n} K_\beta(\zeta_{\pi(a)} - \zeta_{\pi(a+n)})$$

$$K_1(t) = \int_{-1}^{+1} \sin(tx)\, x\, dx$$ O(N)-inv.

Brezin, Hikami (2003)

$$K_4(t) = \int_{-1}^{+1} \sin(tx)\, x^{-1} dx$$ USp(2N)-inv.

Supersymmetry Method

$$\mathrm{Det}^{-1}(E \pm \mathrm{i}\varepsilon - H) = \int \mathrm{e}^{\pm \mathrm{i}(\overline{\varphi},(E \pm \mathrm{i}\varepsilon - H)\varphi)}$$

$$\mathrm{Det}(E - H) = \int \mathrm{e}^{-\mathrm{i}(\overline{\psi},(E-H)\psi)}$$

The GUE_N average $\left\langle \mathrm{e}^{\pm \mathrm{i}(\overline{\varphi},H\varphi) - \mathrm{i}(\overline{\psi},H\psi)} \right\rangle_{\mathrm{GUE}_N}$

depends only on the scalar products

$$M = \begin{pmatrix} (\overline{\varphi},\varphi) & (\overline{\varphi},\psi) \\ (\overline{\psi},\varphi) & (\overline{\psi},\psi) \end{pmatrix}.$$

Switch to the entries of the supermatrix M as the new variables of integration!

Superbosonization

Arrange scalar products $\bar{\varphi}\cdot\varphi$, $\bar{\varphi}\cdot\psi$, $\bar{\psi}\cdot\varphi$, $\bar{\psi}\cdot\psi$

into a supermatrix M.

$$\int F\begin{pmatrix} \bar{\varphi}\cdot\varphi & \bar{\varphi}\cdot\psi \\ \bar{\psi}\cdot\varphi & \bar{\psi}\cdot\psi \end{pmatrix} d\psi\, d\bar{\psi}\;\; d\varphi\, d\bar{\varphi}$$

$$= \int DM\, \mathrm{SDet}^N(M)\, F(M)$$

Superdeterminant:

$$\mathrm{SDet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}(A - BD^{-1}C)}{\mathrm{Det}(D)} = \frac{\mathrm{Det}(A)}{\mathrm{Det}(D - CA^{-1}B)}$$

Supertrace: $\mathrm{STr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Tr}(A) - \mathrm{Tr}(D)$

Universal Construction of Symmetric Superspaces

Complex Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ (Z_2-grading)

(with Cartan involution) $= (\mathfrak{h}_0 + \mathfrak{p}_0) + (\mathfrak{h}_1 + \mathfrak{p}_1)$

Pick real Lie groups $H_0 \subset G_0$ such that $G_0 / H_0 \equiv X$

is globally symmetric Riemannian manifold.

H_0 acts on $\mathfrak{p}_1$ by Ad.

Form associated vector bundle $E = G_0 \times_{H_0} \mathfrak{p}_1 \to X$

$\mathfrak{g}$ canonically acts on 'superfunctions' $\Gamma(X, \wedge E^*)$

Invariant Berezin form $\Gamma(X, \wedge E^*) \xrightarrow{\int_F} \Gamma(X, \wedge^{\mathrm{top}} T^* X)$

Superbosonization

Propn: In the stable range there exists a Berezin form

$$DM : \Gamma(X, \wedge E^*) \to \Gamma(X, \wedge^{\text{top}} T^* X)$$ such that

$$\int d\varphi \, d\overline{\varphi} \int_{\mathrm{F}} \partial_{\psi} \partial_{\overline{\psi}} \; F(M(\overline{\varphi}, \varphi; \overline{\psi}, \psi))$$

$$= \int_X DM \, \mathrm{SDet}^N(M) F(M).$$

Proof: Both sides are distributions (i.e. continuous linear functionals) on $\Gamma(X, \wedge E^*)$. They transform in the same way under the action of the Lie superalgebra $\mathfrak{g}$.

Corollary: the beta=1 version of this proves GOE universality in the non-Gaussian case.